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Boundary Divisors in the Moduli Space of Stable Quintic Surfaces

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BOUNDARY DIVISORS IN THE MODULI SPACE OF STABLE QUINTIC SURFACES

A Dissertation Presented

by

JULIE RANA

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DEDICATION

For Isha and Akash

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ABSTRACT

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My research incorporates several central themes in algebraic geometry, including moduli spaces and their compactifications, singular spaces, and deformation theory. I am especially interested in Gieseker's moduli space $\mathcal{M}_{K^2,\chi}$ of minimal surfaces of general type with fixed numerical invariants, and its Kollár–Shepherd-Barron, Alexeev compactification $\overline{\mathcal{M}}_{K^2,\chi}$. Some of the questions I am interested in include describing which singularities might appear on a stable surface with given invariants, finding concrete models for singular surfaces, and describing the structure of $\overline{\mathcal{M}}_{K^2,\chi}$ along the boundary, especially in the presence of obstructions to $\mathbb{Q}$-Gorenstein deformations of stable surfaces.

In this thesis, I give a bound on which singularities may appear on stable surfaces for a wide range of topological invariants, and use this result to describe all stable numerical quintic surfaces, i.e. stable surfaces with $K^2 = \chi = 5$, whose unique non Du Val singularity is a Wahl singularity. Quintic surfaces are the simplest examples of surfaces of general type and the question of describing their moduli is a long-standing question in algebraic geometry. I then extend the deformation theory of Horikawa in [Hor75] to the log setting in order to describe the boundary divisor of the moduli space $\overline{\mathcal{M}}_{5,5}$ corresponding to

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INTRODUCTION

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In this thesis, I give a bound on which singularities may appear on stable surfaces for a wide range of topological invariants, and use this result to describe all stable numerical quintic surfaces, i.e. stable surfaces with $K^2 = \chi = 5$, whose unique non Du Val singularity is a Wahl singularity. Quintic surfaces are the simplest examples of surfaces of general type and the question of describing their moduli is a long-standing question in algebraic geometry. I then extend the deformation theory of Horikawa in [Hor75] to the log setting in order to describe the boundary divisor of the moduli space $\overline{\mathcal{M}}_{5,5}$ corresponding to these surfaces.

1.1 Background

Algebraic geometry is the study of algebraic varieties, spaces defined as the zero-set of some polynomials. Smooth algebraic varieties over $\mathbb{C}$ can be thought of as complex manifolds that are locally the zero-set of some polynomials. As manifolds, smooth algebraic varieties are isomorphic to a subset of $\mathbb{C}^n$, for some n, called the *dimension* of the variety. For instance, dimension one algebraic varieties are called *curves* and those of dimension two are called *surfaces*. However, algebraic geometry differs from the theory of manifolds in many respects. One important difference is that algebraic geometry allows us to work with singular varieties as well as smooth varieties. Moreover, complex algebraic varieties are endowed with complex structure; two varieties may be the same topologically, but have such different complex structures that they are not isomorphic when viewed as algebraic varieties.

Some of the most important work that algebraic geometers do involves classifying algebraic varieties of a given dimension. Every smooth projective algebraic curve is topologically isomorphic to a compact closed orientable surface, and so may be visualized as a sphere with g handles, where g is an invariant called the genus of the curve. Even as algebraic varieties, every smooth projective curve of genus 0 is isomorphic to the Riemann sphere. But in general two smooth genus g curves are not isomorphic as varieties. Indeed, for curves of genus $g \geq 2$, we can define a *moduli space* $\mathcal{M}_g$, an algebraic space in its own right, in which each point corresponds to a curve of genus g, unique up to automorphisms of the curve. As an algebraic space $\mathcal{M}_g$ has dimension $3g - 3$.

One of the properties of $\mathcal{M}_g$ is that it is not compact. This makes $\mathcal{M}_g$ difficult to work with for several reasons. For example, one desires to know how limits of curves behave "at infinity." That is, we can take limits of smooth curves and obtain something singular that does not correspond to a point in $\mathcal{M}_g$. Another difficulty arises when we try to take intersections of subspaces of $\mathcal{M}_g$. Such intersections are useful, because they can help answer questions about enumerative geometry of curves exhibiting certain types of behavior, for example for computing Gromov-Witten invariants. But defining a reasonable intersection theory on a non-compact space is difficult. This is true even for spaces as simple as $\mathbb{C}^2$, where two lines may not intersect at all. To surmount this, algebraic geometers work instead with compactified spaces. The compactification of $\mathbb{C}^2$ is the projective plane $\mathbb{P}^2$, where lines that were parallel on $\mathbb{C}^2$ now intersect on $\mathbb{P}^2$ "at infinity." Unlike $\mathbb{C}^2$, there are many different and reasonable ways to compactify $\mathcal{M}_g$. Each compactification adds a boundary or wall to the space $\mathcal{M}_g$. The ideal situation is when the new space parametrizes curves with certain types of singularities. For example, the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ contains a divisor (or subspace of dimension one less than that of $\overline{\mathcal{M}}_g$) whose points correspond to irreducible curves with a node [DM69].

Algebraic surfaces have a rough classification, due to Enriques and Kodaira, where surfaces are classified according to an invariant called the Kodaira dimension κ. The Kodaira dimension of an algebraic surface may be either $-\infty$, 0, 1, or 2. Surfaces with $\kappa = -\infty$, 0, or 1 are isomorphic to a surface of one of nine types (for example, K3 surfaces, ruled surfaces, hyperelliptic surfaces, etc.), each of which is well understood. Surfaces of Kodaira dimension 2, known as surfaces *of general type*, can be much more complicated.

Smooth projective surfaces can also be assigned topological invariants K^2, q and p_g, analogous to the genus g of a curve. These invariants may be related in various ways. One important relation is Noether's inequality, which states that every minimal surface of general type satisfies $K^2 \geq 2p_g - 4$. Letting $\chi = 1 - q + p_g$, Gieseker defined, for each pair of possible invariants K^2 and χ, a moduli space $\mathcal{M}_{K^2,\chi}$ whose points correspond to surfaces of general type with the given invariants and mild singularities called Du Val or ADE singularities. One of the pecularities of surfaces of general type is that the moduli spaces $\mathcal{M}_{K^2,\chi}$ can be arbitrarily singular [Vak06] and have many connected components [Cat86]. Moreover, most of the moduli spaces $\mathcal{M}_{K^2,\chi}$ have yet to be explicitly described. That said, there are some very nice complete descriptions of these moduli spaces for surfaces with nice invariants. For instance, for surfaces "on the Noether line" with $K^2 = 2p_g - 4$ and $q = 0$, the moduli spaces $\mathcal{M}_{K^2,\chi}$ were described in detail by Horikawa [Hor76a, Hor76b, Hor78, Hor79, Hor81].

As is the case with $\mathcal{M}_g$, the moduli space $\mathcal{M}_{K^2,\chi}$ is not compact. There are two known compactifications of $\mathcal{M}_{K^2,\chi}$. The better known of these is the Geometric Invariant Theory (GIT) compactification $\overline{\mathcal{M}}^{\mathrm{GIT}}_{K^2,\chi}$, which depends on some extra parameters. The GIT compactification is nice in that gives an answer to the problem of compactifying $\mathcal{M}_{K^2,\chi}$, using a classical construction that has many other important applications (for instance, many of the compactifications of $\mathcal{M}_g$ also involve GIT). However, as a moduli space $\overline{\mathcal{M}}^{\mathrm{GIT}}_{K^2,\chi}$ is not ideal. One reason is that even in simple cases, it is difficult to describe all types of singularities that GIT semistable surfaces may have. Moreover, many semistable points correspond to surfaces with singularities that are in some sense too degenerate.

More recently, Kollár, Shepherd-Barron, and Alexeev constructed a different compactification $\overline{\mathcal{M}}_{K^2,\chi}$, called the KSBA compactification of $\mathcal{M}_{K^2,\chi}$ [KSB88, Ale94]. As an algebraic space $\overline{\mathcal{M}}_{K^2,\chi}$ is much more complicated than $\overline{\mathcal{M}}^{\mathrm{GIT}}_{K^2,\chi}$. However, as a moduli space, $\overline{\mathcal{M}}_{K^2,\chi}$ is more natural. Its points parametrize so-called *stable surfaces*, i.e., surfaces with ample canonical class and semi log canonical singularities. Semi log canonical singularities are completely classified in [KSB88] and include a number of types of isolated singularities, one type of which are the cyclic quotient singularities described below. They also include orbifold double normal crossing singularities, which are locally analytically isomorphic to two surfaces intersecting transversally or in a curve that contains mild singularities on each component surface.

In studying the moduli space $\overline{\mathcal{M}}_{K^2,\chi}$, a natural question to ask is which types of singularities actually appear on stable surfaces with given invariants. Even better, can we describe loci in $\overline{\mathcal{M}}_{K^2,\chi}$ whose points correspond to surfaces with a given type of singularity? Two types of semi log canonical singularities in particular typically correspond to divisors in the boundary of the moduli space $\overline{\mathcal{M}}_{K^2,\chi}$, for example in the absence of obstructions to deformations. One type are the orbifold double normal crossing singularities described above. Given a surface with orbifold double normal crossing singularities, there is a condition on the curve of intersection of the two components which guarantees that the singularity has a local one-parameter $\mathbb{Q}$-Gorenstein smoothing. As long as this smoothing is unobstructed, the equisingular deformations of the surface will give a generically smooth divisor in the boundary of $\overline{\mathcal{M}}_{K^2,\chi}$.

Surfaces with *cyclic quotient singularities* are also expected to give a divisor in $\overline{\mathcal{M}}_{K^2,\chi}$. A cyclic quotient singularity of type $\frac{1}{n}(1,a)$, where a and r are relatively prime, is locally analytically isomorphic to a quotient of $\mathbb{C}^2$ by the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Explicitly, the action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{C}^2$ is given by $(x,y) \mapsto (x, \zeta^a y)$, where ζ is a primitive n^{th} root of unity. An important subset of cyclic quotient singularities are those that admit at least a one-parameter $\mathbb{Q}$-Gorenstein smoothing, because only surfaces with these singularities may actually occur in the boundary of $\overline{\mathcal{M}}_{K^2,\chi}$ as stable limits of families of surfaces in $\mathcal{M}_{K^2,\chi}$. Of these, the ones that admit at most a one-parameter smoothing are called *Wahl singularities*, and are the cyclic quotient singularities of type $\frac{1}{n^2}(1, na-1)$ where a and n are relatively prime. As long as there are no obstructions to deformations, surfaces whose unique non Du Val (or ADE) singularity is a Wahl singularity will give divisors in $\overline{\mathcal{M}}_{K^2,\chi}$, corresponding to equisingular deformations of these surfaces.

One method to find surfaces with Wahl singularities is to try to explicitly construct them by constructing their minimal resolutions, which are smooth and contain explicit strings of rational curves with negative self-intersections. This method has been met with some success, most notably by Y. Lee, H. Park, J. Park, and D. Shin with their constructions of surfaces of general type with $p_g = q = 0$ found in [LP07, PPS09b, PPS11, PPS09a]. I use this method, together with bounds on which Wahl singularities may appear on surfaces with certain invariants, to find and describe the divisor in $\overline{\mathcal{M}}_{5,5}$ corresponding to surfaces whose unique non Du Val singularity is a Wahl singularity.

1.2 Stable numerical quintic surfaces

The simplest examples of surfaces of general type are quintic surfaces, or surfaces in $\mathbb{P}^3$ which are the zero-set of a polynomial of degree 5. The moduli space $\mathcal{M}_{5,5}$ of numerical quintic surfaces, or surfaces with the same invariants as quintic surfaces, was described by Horikawa in [Hor75]. This moduli space is a union of two 40-dimensional irreducible components meeting, transversally at a general point, in a 39-dimensional irreducible variety. Figure 1 gives a schematic diagram of $\mathcal{M}_{5,5}$. We should remark that each component parametrizes *smooth* surfaces with $K^2 = \chi = 5$, although surfaces in components IIa and IIb are not quintic surfaces in the usual sense.

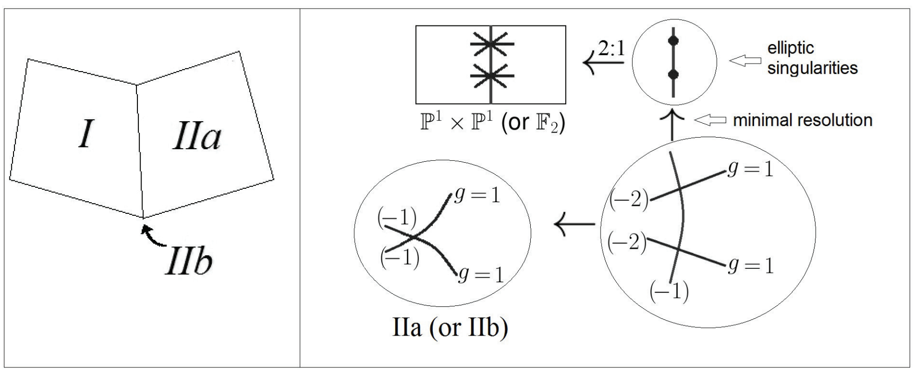


Figure 1. On the left, a visualization of $\mathcal{M}_{5,5}$. Components I and IIa are 40-dimensional; IIb is 39-dimensional. On the right, the construction of a numerical quintic surface of type IIa (or IIb) from a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ (or $\mathbb{F}_2$).

I am interested in describing which types of singularities may appear on a stable numerical quintic surface. As discussed above, the first natural singularities to look at are Wahl singularities. In what follows, let X be a stable surface whose unique non Du Val singularity is a Wahl singularity, and let $\tilde{X}$ be its minimal resolution. Let $\bar{X}$ be the minimal model of X, obtained by contracting all (-1) curves on $\tilde{X}$.

Lemma 1.1. If K_X and $K_{\bar{X}}$ are big and nef, then $K_X^2 > K_{\bar{X}}^2$.

Lemma 1.1 is similar to a result of Kawamata [Kaw92, 2.4, 4.6], but in his case the surface X must be the central fiber of a $\mathbb{Q}$-Gorenstein degeneration whose generic fiber is a smooth connected surface. I study the case where the difference $K_X^2 - K_{\bar{X}}^2$ is as small as possible: What happens when $K_X^2 = K_{\bar{X}}^2 + 1$?

The minimal resolution $\tilde{X}$ of a surface X with a Wahl singularity contains a string of exceptional curves with negative self-intersections, called the T-string of the singularity. If the T-string of a certain singularity contains r exceptional curves, then we say that the singularity has *length* r. It is tempting to try to bound the type of Wahl singularity that may appear on a given surface by bounding its length. In fact, this is possible; in [Lee99], Y. Lee shows that if X has a unique Wahl singularity of length r and at most Du Val singularities otherwise, then $r \leq 400K_{\tilde{X}}^2$. The following result greatly improves Lee's bound, although it applies only to those surfaces for which $K_X^2 = K_{\bar{X}}^2 + 1$.

Theorem 1.2. Let X be a surface with a unique Wahl singularity p of length r and at most Du Val singularities elsewhere, let $\tilde{X}$ be its minimal resolution, and $\bar{X}$ the minimal model of $\tilde{X}$. If K_X and $K_{\tilde{X}}$ are big and nef and if $K_{\bar{X}}^2 = K_X^2 - 1$, then $r = 1$ or 2. That is, p is a $\frac{1}{4}(1,1)$, $\frac{1}{9}(1,2)$, or $\frac{1}{9}(1,5)$ singularity.

Using Horikawa's descriptions of surfaces lying on the Noether line [Hor76a], I can improve the result further for surfaces near it:

Theorem 1.3. With the same hypotheses as in Theorem 1.2, assume moreover that $K_X^2 = 2p_g - 3$. If $\bar{X}$ is of general type then p is a $\frac{1}{4}(1,1)$ singularity. Moreover, if p is a $\frac{1}{4}(1,1)$ singularity and $K_X^2 > 3$, then $\bar{X}$ is of general type.

Theorem 1.3 suggests that it is possible to describe all stable surfaces lying one above the Noether line whose unique non Du Val singularity is a $\frac{1}{4}(1,1)$ singularity. In this thesis, I do this for the case of stable numerical quintic surfaces by looking at the minimal resolution $\tilde{X}$ of a stable numerical quintic surface X. I prove that the surface $\tilde{X}$, which contains a rational curve of self-intersection -4, arises from the double cover of a smooth or nodal quadric, with branch locus intersecting a given curve in one of a few specified ways.

There are a few examples of stable numerical quintic surfaces with a unique $\frac{1}{4}(1,1)$ singularity that correspond to 38- and 39-dimensional loci in $\overline{\mathcal{M}}_{5,5}$. Three important ones are surfaces of types 1, 2a, and 2b. To construct a surface of type 1, take a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$, branched over a sextic intersecting a given diagonal tangentially at 6 points. The preimage of the diagonal is two (-4)-curves, intersecting at 6 points. Contracting one of these (-4)-curves gives a stable numerical quintic surface of type 1. The other important examples are surfaces of types 2a or 2b, the minimal resolutions of which arise from double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ or a quadric cone, respectively. The branch curve of this double cover is a sextic B intersecting a given fiber at two nodes of B and transversally at two other points.

We remark that Friedman [Fri83] raises the question of describing deformations of 2b surfaces. Theorem 1.4 answers this question.

Surfaces of types 1 and 2a in particular correspond to 39-dimensional loci in $\overline{\mathcal{M}}_{5,5}$. For these, I prove vanishing of the cohomology group in which obstructions to deformations lie, and conclude that the closures of these loci give generically smooth Cartier divisors in $\overline{\mathcal{M}}_{5,5}$. Obstructions to deformations of 2b surfaces do not vanish. By extending the deformation theory of Horikawa to log pairs, I prove in this thesis the following theorem.

Theorem 1.4. The locus of stable numerical quintic surfaces whose unique non Du Val singularity is a $\frac{1}{4}(1,1)$ singularity forms a divisor in $\overline{\mathcal{M}}_{5,5}$ which consists of two 39-dimensional components $\overline{1}$ and $\overline{2\mathrm{a}}$ meeting, transversally at a general point, in a 38-dimensional component $\overline{2\mathrm{b}}$. This divisor is Cartier at general points of the $\overline{1}$, $\overline{2\mathrm{a}}$, and $\overline{2\mathrm{b}}$ components. These components are the closures of the loci of 1, 2a, and 2b surfaces described above. Moreover, the type $\overline{1}$, $\overline{2\mathrm{a}}$, and $\overline{2\mathrm{b}}$ components belong to the closure of the components in $\mathcal{M}_{5,5}$ of types I, IIa, and IIb, respectively.

The idea of the proof of Theorem 1.4 is as follows. I begin by showing that the space of obstructions to $\mathbb{Q}$-Gorenstein deformations of a 2b surface is one-dimensional. Therefore, the moduli space of equisingular deformations of 2b surfaces is a hypersurface in some ambient space. I then locate a subfunctor of the functor of $\mathbb{Q}$-Gorenstein deformations, corresponding to deformations of covers, and show that these deformations are unobstructed. This shows that there is a smooth component in the moduli space of equisingular deformations of a 2b surface. This observation implies that it is enough to show that the second order part of the Kuranishi function, given by the Schouten bracket, does not vanish and is not a square. I describe this bracket by extending the deformation theory of Horikawa in [Hor75].

As described above, another type of singularity to consider are the orbifold double normal crossing singularities. It is not obvious how to construct a family of stable numerical quintic surfaces whose special fiber has orbifold double normal crossing singularities. I use a construction of Dolgachev in [Dol96] involving certain weighted homogeneous singularities, called Fuchsian singularities, to construct surfaces with orbifold double normal crossing singularities. To construct a Fuchsian singularity, take a tiling of the upper half plane $\mathbb{H}$ by a polygon with angles $\frac{\pi}{p_1}, \frac{\pi}{p_2}, \cdots, \frac{\pi}{p_r}$. Let Γ^+ be the group of orientation-preserving isometries of this tiling. Contracting the zero-section of the cotangent bundle $\Omega^1(\mathbb{H}/\Gamma^+) \to \mathbb{H}/\Gamma^+$ produces the desired singularity. See Figure 2 for a visualization.

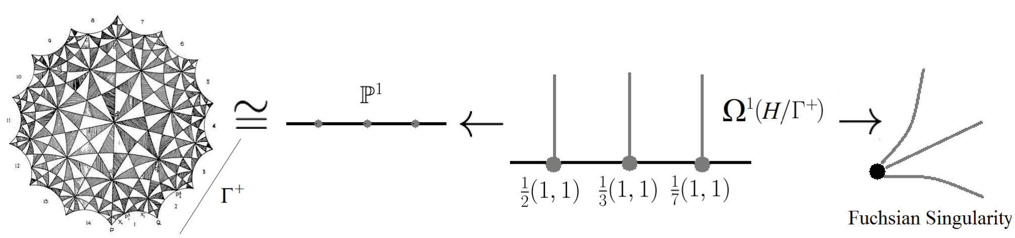


Figure 2. The weighted homogeneous singularity K_{12}. The triangular tiling has angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$.

Fuchsian singularities are not semi log canonical. However, for the 22 Fuchsian hypersurface singularities, we can obtain the stable limit of a smoothing by performing a certain weighted blowup, described by Dolgachev [Dol96]. This results in a surface with orbifold double normal crossing singularities, one component of which is a K3 surface containing specific Du Val singularities. I have done this for a few examples, and expect to prove the following conjecture in the coming months.

Conjecture 1.5. Each of the 22 exceptional Fuchsian singularities corresponds to a generically Cartier divisor in $\overline{\mathcal{M}}_{5,5}$.

I should remark that P. Gallardo proved part of this conjecture independently in his thesis [Gal], using different methods. To show smoothness, he uses a theorem of Shustin and Tyomkin [ES99]. In the coming months, I hope to prove Conjecture 1.5 more directly by showing that surfaces obtained from smoothings of Fuchsian singularities and containing a K3 component as above have unobstructed $\mathbb{Q}$-Gorenstein deformations.

1.3 Future directions

I would like to construct more examples of surfaces lying one above the Noether line containing a unique $\frac{1}{4}(1,1)$ singularity. I expect that most of these constructions will be similar to the examples of degenerations of numerical quintic surfaces described in this thesis. As a first step, this will likely involve describing the minimal resolutions. For instance, if the minimal resolutions are themselves double covers, as is often the case with surfaces on the Noether line, then one can describe the image of any (-4)-curve under this double cover. A part of this classification, especially describing how the branch divisors of such maps must intersect certain curves, would be a great project for interested undergraduates, perhaps in the form of an REU. Armed with such examples, I would like to extend Horikawa's deformation theory and Theorem 1.4 to prove the following conjecture.

Conjecture 1.6. The locus of stable surfaces with $K^2 = 2p_g - 3$ and $q = 0$ whose unique non Du Val singularity is a Wahl singularity of type $\frac{1}{4}(1,1)$ forms a generically smooth Cartier divisor in $\overline{\mathcal{M}}_{K^2,\chi}$.

Another possible direction is to consider what happens when the minimal model is not of general type. Then the minimal model is an elliptic surface with a certain configuration of singular curves. It would be interesting to try to construct elliptic surfaces with these configurations.

I would also like to find more examples of surfaces with orbifold double normal crossing singularities, and extend Conjecture 1.5 to other surfaces of general type.

As a final note, an interesting problem in algebraic geometry is the question of extending the Hassett-Keel program from curves to surfaces. The philosophy of the Hassett-Keel program for curves is that many compactifications of $\mathcal{M}_g$ are related by divisorial contractions and flips, and that these birational maps have a modular interpretation. One can ask similar questions for surfaces. For instance, how are the compactifications $\overline{\mathcal{M}}_{5,5}$ and $\overline{\mathcal{M}}_{5,5}^{\mathrm{GIT}}$ related? In his thesis, P. Gallardo [Gal] considers this question from the point of view of $\overline{\mathcal{M}}_{5,5}^{\mathrm{GIT}}$, by finding the semistable replacement of certain singularities. From the other side of things, we can ask which boundary divisors on $\overline{\mathcal{M}}_{5,5}$ are contractible? Once contracted, is there a way to interpret the resulting space as a moduli space?

CHAPTER 2

RESTRICTIONS ON SINGULARITIES

We give bounds on which Wahl singularities may appear on a stable surface with limited invariants.

The two-dimensional quotient singularities which admit $\mathbb{Q}$-Gorenstein smoothings are called T-singularities, and are those cyclic quotient singularities of the form $\frac{1}{dn^2}(1, dna - 1)$ where a and n are coprime [KSB88]. Those which admit only a one-parameter $\mathbb{Q}$-Gorenstein smoothing are T-singularities with $d = 1$. They were studied first by Wahl [Wah81] and so are called *Wahl singularities*.

The minimal resolution of a surface with a Wahl singularity of the form $\frac{1}{n^2}(1, na - 1)$ contains a string of exceptional curves $C_1, \ldots, C_r$ such that

$$C_i \cdot C_j = \begin{cases} 1 & \text{if } i = j \pm 1 \\ -b_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

where $[b_1, \cdots, b_r]$ is the Hirzebruch-Jung continued fraction expansion of $\frac{n^2}{na-1}$. We say that the T-string $C_1, \ldots, C_r$ and the singularity corresponding to it have *length* r.

The T-string of a Wahl singularity has an especially useful iterative description by Wahl.

Proposition 2.1. [Wah81] The cyclic quotient singularity $\frac{1}{4}(1, 1)$ is a Wahl singularity of length 1 with $b_1 = 4$. Moreover, every Wahl singularity has a T-string $C_1, \ldots, C_r$ where $[b_1, \cdots, b_r]$ is one of the following types:

i) if $[b_1, \ldots, b_{r-1}]$ is a Wahl singularity then

$$[2, b_1, \ldots, b_{r-1} + 1]$$

and

$$[b_1 + 1, b_2, \ldots, b_{r-1}, 2]$$

are also Wahl singularities and

ii) The T-string of any Wahl singularity may be found by starting with the resolution [4] and iterating the steps described in *i*).

Because they are quotient singularities, Wahl singularities are log terminal. Thus, if W contains a unique Wahl singularity and is otherwise smooth, and if $\phi : X \to W$ is its minimal resolution containing the T-string $C_1, \ldots, C_r$, then we can write

$$K_X = \phi^* K_W + \sum_{i=1}^{r} a_i C_i$$

where$-1 < a_i < 0$. There is a very simple relationship between K_X^2 and K_W^2, also discovered by Wahl.

Lemma 2.2. [Wah81] Let W be a surface with a unique Wahl singularity of length r and possibly Du Val singularities and let X be is its minimal resolution. Then $K_X^2 = K_W^2 - r$.

To describe the possible Wahl singularities which may occur on a surface with given invariants, one might hope to bound r in terms of K_W^2 and K_S^2, where S is the minimal model of X. The best known bound to date was discovered by Y. Lee.

Theorem 2.3. [Lee99, Th. 23] Suppose W is a surface of general type with a unique Wahl singularity of length r. Let X be its minimal resolution and S the minimal model of X. If K_S is ample then $r \leq 400(K_S^2)^4$.

We prove a much nicer bound, at the cost of restricting to a much smaller class of surfaces.

Let W be a surface with a unique Wahl singularity of length r and possibly Du Val singularities, let $\phi : X \to W$ be its minimal resolution, and $\pi : X \to S$ be the minimal model of X as in Figure 5. If π contracts n (-1)-curves, then $K_X^2 = K_S^2 - n$. By Lemma 2.2, we have $K_X^2 = K_W^2 - r$. We hope to bound r by investigating the relationship between n and r. The following Lemma shows that if K_W and K_S are big and nef, then $r > n$; that is, $K_W^2 > K_S^2$.

Lemma 2.4. If K_W and K_S are big and nef then $K_W^2 > K_S^2$.

Proof. Let W be a surface with a unique Wahl singularity of type $\frac{1}{n^2}(1, na-1)$ at p and at most Du Val singularities elsewhere. Since resolving the Du Val singularities on W does not affect K_W^2 and nefness of K_W, we can assume without loss of general that W is smooth away from p. Choose $m > 0$ such that $n|m$. Then mK_W is Cartier.

Since K_S and K_W are big and nef, we have

$$h^i(S, mK_S) = h^i(S, (m-1)K_S + K_S) = 0$$

and

$$h^i(W, mK_W) = h^i(W, (m-1)K_W + K_W) = 0$$

for $i > 0$ by Kawamata–Viehweg vanishing. In particular,

$$\chi(S, mK_S) = h^0(S, mK_S) \text{ and } \chi(W, mK_W) = h^0(W, mK_W).$$

We claim that

$$h^0(W, mK_W) > h^0(X, mK_X) = h^0(S, mK_S)$$

for m sufficiently large. To see this, write

$$K_X = \phi^*(K_W) + \sum_i a_i C_i,$$

where $-1 < a_i < 0$, because p is log terminal. Choose m sufficiently large and divisible so that the denominators of the a_i divide m for all i. Then

$$\phi^*(mK_W) = mK_X + C,$$

where $C = -m\sum_i a_i C_i$ is an effective Cartier divisor. Consider the restriction exact sequence

$$0 \to \mathcal{O}(mK_X) \to \mathcal{O}(\phi^*(mK_W)) \to \mathcal{O}_C \to 0.$$

To show that $h^0(W, mK_W) > h^0(X, mK_X)$, it suffices to show that the induced map

$$H^0(X, \phi^*(mK_W)) \to H^0(C, \mathcal{O}_C)$$

is nonzero. By the Kawamata-Shokurov base point free theorem, we can choose a section s of mK_W, for m sufficiently large and divisible, such that $s(p) \neq 0$. Thus, the map is indeed nonzero.

Since p has index n, the divisor mK_W is Cartier and the usual Riemann–Roch Theorem holds [Rei97]. Thus,

$$\begin{aligned}
\chi(W, \mathcal{O}_W) + \frac{m(m-1)}{2}K_W^2 &= \chi(W, mK_W) \\
&= h^0(W, mK_W) \\
&> h^0(S, mK_S) \\
&= \chi(S, mK_S) \\
&= \chi(S, \mathcal{O}_S) + \frac{m(m-1)}{2}K_S^2.
\end{aligned}$$

Since ψ is the resolution of a rational singularity, we have

$$\chi(W, \mathcal{O}_W) = \chi(X, \mathcal{O}_X) = \chi(S, \mathcal{O}_S),$$

and so $K_W^2 > K_S^2$ as we wished to show. □

Remark 2.5. Kawamata makes a similar statement, but requires that W be the central fiber of a $\mathbb{Q}$-Gorenstein degeneration $\mathscr{X} \to \Delta$ whose generic fiber is a smooth connected surface [Kaw92, 2.4, 4.6].

Because it is difficult to give a useful bound on r without any assumptions on n, we begin by restricting to the case that $K_W^2 = K_S^2 + 1$. We will then use Noether's inequality together with Lemma 2.4 to show that this holds in the case that W is a stable numerical quintic surface.

Theorem 2.6. Suppose W is a surface with a unique Wahl singularity p of length r and at most Du Val singularities elsewhere. Let X be its minimal resolution, and $\pi : X \to S$ the minimal model of X as in Figure 5. Suppose that $K_W^2 = K_S^2 + 1$. If K_W and K_S are big and nef, then p is a $\frac{1}{4}(1,1)$, $\frac{1}{9}(1,2)$, or $\frac{1}{9}(1,5)$ singularity.

The proof of Theorem 2.6 requires two lemmas, but we begin with some notation.

Let us write π as a composition of birational maps, each of which contracts a single (-1)-curve to a point $x_j \in X_j$:

$$X = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = S$$

For $j \in \{1, \ldots, n\}$, let $F_j = \pi_j^{-1}(x_{j-1}) \subset X_j$ be the (-1)-curve on X_{j-1} obtained by blowing up the smooth point $x_{j-1} \in X_{j-1}$. Let

$$E_j = (\pi_j \circ \pi_{j+1} \circ \cdots \circ \pi_n)^{-1}(x_{j-1}) \subset X.$$

We call each E_j an "exceptional divisor" of π. With this notation, we can write

$$K_X = \pi^*(K_S) + \sum_{i=1}^{n} E_j.$$

We note that because the maps π_i are birational, the self-intersection of E_j is (-1) and $E_i \cdot E_j = 0$ for $i \neq j$. We have $E_n = F$ for some (-1)-curve F. Moreover, each E_j contains at least one (-1)-curve and E_j is not necessarily reduced, but its reduction is a tree of rational curves. Finally, each E_j contains no loops of curves and pairs of curves in E_j intersect at most once.

Lemma 2.7. $\sum_{j=1}^{n} \sum_{i=1}^{r} E_j \cdot C_i \leq r$.

Proof. By adjunction

$$K_X \cdot \sum_{i=1}^{r} C_i = \sum_{i=1}^{r} (b_i - 2).$$

We claim that

$$\sum_{i=1}^{r}(b_i - 2) = r + 1. \tag{2.1}$$

The proof will be by induction on r. If $r = 1$ then the T-string consists of a single (-4)-curve, and we have $b_1 - 2 = 4 - 2 = 2$.

Now suppose that for any T-string of length k we have $\sum_{i=1}^{k}(b_i - 2) = k + 1$. By Proposition 2.1 any T-string of length $k+1$ is of the form $\{C_1, \cdots, C_{k+1}\}$ with $C_i^2 = -b_i$ such that $[b_1, \ldots, b_{k+1}]$ is either

1. $[2, b_1', \ldots, b_k' + 1]$ or

2. $[b_1' + 1, b_2', \ldots, b_k', 2]$,

where $[b_1', \ldots, b_k']$ corresponds to a T-string $C_1', \ldots, C_k'$ of length k. Thus

$$\sum_{i=1}^{k+1}(b_i - 2) = \sum_{i=1}^{k}(b_i' - 2) + 1 = k + 2,$$

proving the claim.

Since K_S is nef, we have

$$\pi^* K_S \cdot \sum_{i=1}^{r} C_i \geq 1.$$

Therefore,

$$K_X \cdot \sum_{i=1}^{r} C_i = \sum_{i=1}^{r}(\pi^* K_S + \sum_{j=1}^{n} E_j) \cdot C_i \geq 1 + \sum_{i=1}^{r}\sum_{j=1}^{n} E_j \cdot C_i \tag{2.2}$$

and so

$$\sum_{i=1}^{r}\sum_{j=1}^{n} E_j \cdot C_i \leq K_X \cdot \sum_{i=1}^{r} C_i - 1.$$

Combining this with Equation (2.1) gives

$$\begin{aligned}\sum_{i=1}^{r}\sum_{j=1}^{n} E_j \cdot C_i &\leq \sum_{i=1}^{r} C_i \cdot K_X - 1 \\ &= \sum_{i=1}^{r}(b_i - 2) - 1 \\ &= r + 1 - 1 = r.\end{aligned}$$

□

Lemma 2.8. $\sum_{i=1}^{r}\sum_{j=1}^{n} E_j \cdot C_i \geq 2n$.

Proof. The claim is obvious for $n = 0$. Fix an exceptional divisor $E = E_j$ for some j and a curve $C = C_i$ for some i. If $C \subset E$, then $C \cdot E_j = -1$ if and only if

$$(\pi_j \circ \pi_{j+1} \circ \cdots \circ \pi_n)(C) = x_j$$

and

$$(\pi_{j+1} \circ \pi_{j+2} \circ \cdots \circ \pi_n)(C) = F_j.$$

Otherwise, $C \cdot E_j = 0$. Thus, $\sum_{i=1}^{r} C_i \cdot E \geq -1$. Since we want $\sum_{i=1}^{r} C_i \cdot E \geq 2$, it suffices to show that there are at least three points of intersection (counted with multiplicity) among curves in the T-string which are not in E and curves in E.

Given a T-string $\mathcal{C}$ containing curves $C_1, \ldots, C_r$, let

$$\bullet \text{ ---- } \bullet \text{ ---- } \cdots \text{ ---- } \bullet \text{ ---- } \bullet$$

be the dual graph of the T-string, where the i^{th} vertex corresponds to the curve C_i. If $C_i \subset E$, we replace the i^{th} vertex in the above graph by a box, and denote the resulting graph by Γ_E. For instance, if Γ_E is

$$\square \text{ ---- } \bullet \text{ ---- } \square \text{ ---- } \bullet \text{ ---- } \square$$

then there are at least 4 points of intersection among curves in $\mathcal{C} \backslash E$ and curves in E. With this notation we can immediately see that if there are less than 3 such intersections then Γ_E must have one of the following forms:

1)

$$\bullet \text{ ---- } \cdots \text{ ---- } \bullet \text{ ---- } \square \text{ ---- } \cdots \text{ ---- } \square \text{ ---- } \bullet \text{ ---- } \cdots \text{ ---- } \bullet$$

2)

$$\square \text{ ---- } \cdots \text{ ---- } \square \text{ ---- } \bullet \text{ ---- } \cdots \text{ ---- } \bullet \text{ ---- } \square \text{ ---- } \cdots \text{ ---- } \square$$

3)

$$\square \text{ --- } \cdots \text{ --- } \square \text{ --- } \bullet \text{ --- } \cdots \text{ --- } \bullet$$

or

$$\bullet \text{ --- } \cdots \text{ --- } \bullet \text{ --- } \square \text{ --- } \cdots \text{ --- } \square$$

Since $n \geq 1$, there is a (-1)-curve F in E. Because $C_i^2 < -1$ for all i, we also have that $C_i \cdot F \geq 0$ for each i. We claim moreover that $\phi^* K_W \cdot F > 0$. Suppose for a contradiction that $\phi^* K_W \cdot F \leq 0$. Since K_W is nef, this implies that $\phi_W^K \cdot F = 0$. The surface W is a resolution of Du Val singularities on a stable surface W'. Let $\theta : W \to W'$ be the resolution of Du Val singularities. Since $K_{W'}$ is ample, this implies that F is contracted by θ. But then F is a (-2) curve, a contradiction.

Writing $K_X = \phi^* K_W + \sum_{i=1}^r a_i C_i$, we have

$$\begin{aligned} \sum_{i=1}^r C_i \cdot F &\geq -\sum_{i=1}^r a_i C_i \cdot F \\ &= \phi^* K_W \cdot F - K_X \cdot F \\ &= \phi^* K_W \cdot F + 1 \\ &> 1. \end{aligned}$$

In particular,

$$\sum_{i=1}^r C_i \cdot F \geq 2. \tag{2.3}$$

Thus F intersects at least two of the curves C_i, or one curve C_i with multiplicity at least two. Moreover, if a curve C_i intersecting F is contained in E, then $\pi_{k+1} \circ \cdots \circ \pi_n(C_i) = F_k$ for some k. Thus, $\pi_{k+1} \circ \cdots \circ \pi_n(C_i)$ is a smooth curve and so $C_i \cdot F = 1$. Because E does not contain loops of curves, we see that in Cases 1 and 3 the curve F must intersect at least one C_i which is not in E. In Case 1, this gives our third point of intersection. In Case 3 it gives a second.

We now have only to deal with Cases 2 and 3, for both of which we now have

$$\sum_{i=1}^r C_i \cdot E \geq 1.$$

Suppose there are k exceptional curves E such that $\sum_{i=1}^{r} C_i \cdot E = 1$. We claim that $k = 0$.

Suppose for a contradiction that $k > 0$. By the above argument and Lemma 2.7 we have

$$r \geq \sum_{j=1}^{n} \sum_{i=1}^{r} E_j \cdot C_i \geq 2(n-k) + k = 2n - k.$$

Since $r = n + 1$, we have that $k \geq n - 1$. On the other hand, since $E_n = F$ is a single (-1)-curve, we have $k \leq n - 1$. Thus, $k = n - 1 = r - 2$. In particular, this implies that $r \geq 3$, that all but two curves in $\mathcal{C}$ are contained in exceptional divisors, and that all exceptional divisors other than E_n satisfy $\sum_{i=1}^{r} C_i \cdot E = 1$. This means that there is only one (-1) curve which must therefore be contained in all of the exceptional divisors.

Let us begin with Case 2. If the (-1)-curve F intersects both a bullet and a box in Γ_{E_1}, then since Γ_{E_i} is obtained from Γ_{E_1} by replacing some boxes with bullets, this gives the third intersection point for all E_i. So we can assume that it intersects two boxes as in Figure 3.

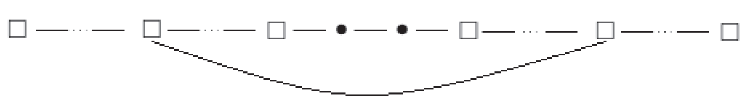


Figure 3. Γ_{E_1}. The curved line along the bottom represents the (-1)-curve F.

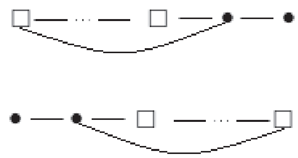
Every exceptional divisor E_j other than $E_n = F$ satisfies $\sum_{i=1}^{r} C_i \cdot E_j = 1$ and must be a subset of E_1. Each E_j also contains F, so the only possibility is that F intersects C_1 and C_r. However, by [Kaw92, 3.2] we have $a_1 + a_r = -1$, so

$$-1 = K_X \cdot F = (\phi^* K_W + \sum_{i=1}^{r} a_i C_i) \cdot F = \phi^* K_W \cdot F - 1.$$

Therefore, $K_W \cdot \phi(F) = 0$. Since $\phi(F)$ has positive arithmetic genus and K_W is nef, this is a contradiction.

The final case to consider is Case 3. Here Γ_{E_1} must be of the form:

or



where the curved line along the bottom represents the (-1)-curve F. Here, E_1 is a chain of curves with a (-1)-curve at the end. Contracting F under π_n gives another (-1)-curve, and so C_r is necessarily a (-2)-curve. Contracting $\pi_n(C_r)$ under π_{n-1} must also give a (-1)-curve, so that C_{r-1} must also be a (-2)-curve. Continuing in this way, we see that E_1 must consist of $n-1$ (-2)-curves and a (-1)-curve F. Thus, $\mathcal{C}$ must correspond to the Wahl singularity with Hirzebruch-Jung continued fraction $[r+3, 2, \ldots, 2]$, $[2, \ldots, 2, 2, r+3]$, $[r, 5, 2, \ldots, 2]$ or $[2, \ldots, 2, 5, r]$. Without loss of generality, we need only consider the cases $[r+3, 2, \ldots, 2]$ and $[r, 5, 2, \ldots, 2]$.

Suppose first that $b_2 = 2$. Then using the fact that K_S is nef and that $C_1 \cdot F \geq 1$, we have

$$0 = K_X \cdot C_2 = \pi^* K_S \cdot C_2 + \sum_{j=1}^{n} E_j \cdot C_2 \geq \pi^* K_S \cdot C_2 + 1 \geq 1$$

and we have a contradiction.

The only Wahl singularity left to consider is that with Hirzebruch-Jung continued fraction $[r, 5, 2, \ldots, 2]$. In this case, Γ_{E_1} together with F is the graph shown in Figure 4.

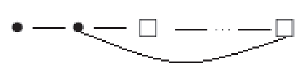


Figure 4. The remaining possibility for Γ_{E_1}.

Since $K_X \cdot C_2 = 3$ and $C_2 \cdot \sum_{j=1}^{n} E_j \geq n$ we have

$$\begin{aligned} 0 &\leq \pi^* K_S \cdot C_2 \\ &= (K_X - \sum_{j=1}^{n} E_j) \cdot C_2 \\ &= 3 - \sum_{i=1}^{n} E_j \cdot C_2 \\ &= 3 - n. \end{aligned}$$

This gives $n \leq 3$, and so $r \leq 4$. If $r = 3$, then $C_2^2 = -5$. The image $\pi(C_2)$ has self-intersection 0 and arithmetic genus 1. Therefore, by adjunction $K_S \cdot \pi(C_2) = 0$, contradicting the fact that K_S is big and nef.

Similarly, if $r = 4$ then the $\pi(C_2)$ has self-intersection 1 and arithmetic genus 1. By adjunction, we have $K_S \cdot \pi(C_2) = -1$, contradicting the fact that K_S is nef.

Since all possibilities lead to a contradiction, we conclude that $k = 0$. $\square$

We can now prove Theorem 2.6.

Proof of Theorem 2.6. We must show that $r \leq 2$. By Lemma 2.7 we have

$$\sum_{j=1}^{n} \sum_{i=1}^{r} E_j \cdot C_i \leq r.$$

On the other hand, Lemma 2.8 tells us that

$$\sum_{j=1}^{n} \sum_{i=1}^{r} E_j \cdot C_i \geq 2n.$$

Since $n = r - 1$, we have that $r \leq 2$, so p is a $\frac{1}{4}(1,1)$, $\frac{1}{9}(1,2)$, or $\frac{1}{9}(1,5)$ singularity. $\square$

Now suppose that W be a stable surface whose unique non Du Val singularity is a Wahl singularity p of length r. Let $\phi : X \to W$ be the minimal resolution of W, and let $\pi : X \to S$ be the minimal model of W, which is obtained from X by contracting n (-1)-curves.

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[dr, "\pi"] & \\ W & & S \end{tikzcd}$$

Figure 5. The surfaces W, X, and S.

Theorem 2.9. Suppose that K_W is big and nef and satisfies $K_W^2 = 2p_g - 3$. If S is of general type then p is a $\frac{1}{4}(1,1)$ singularity. Moreover, if p is a $\frac{1}{4}(1,1)$ singularity and $K_W^2 > 3$, then S is of general type.

We remark that Noether's inequality (that for surfaces S of general type, we have $K_S^2 \geq 2p_g - 4$) implies the following corollary of Lemma 2.4.

Corollary 2.10. If the surface W satisfies $K_W^2 = 2p_g - 4$, then S is not of general type.

The significance of the equality $K_W^2 = 2p_g - 3$ in Theorem 2.9 is that such surfaces lie one above the "Noether line" $K_W^2 = 2p_g - 4$. That is, this K_W^2 is the smallest it can be and still have S be of general type.

For the proof of Theorem 2.9, we recall Horikawa's description of minimal surfaces of general type with $K^2 = 2p_g - 4$ in [Hor76a]. For $d \geq 0$, the Hirzebruch surface $\mathbb{F}_d$ is the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ whose zero section Δ_0 has self-intersection $-d$. We denote by Γ a generic fiber of $\mathbb{F}_d$ and note that $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 2.11. [Hor76a] Let S be a minimal algebraic surface with $K^2 = 2p_g - 4$ for $p_g \geq 3$. Then S is the minimal resolution of one of either:

1. ($K^2 = 2$) a double cover of $\mathbb{P}^2$ branched over a curve of degree 8,
2. ($K^2 = 8$) a double cover of $\mathbb{P}^2$ branched over a curve of degree 10,
3. a double cover of $\mathbb{F}_d$, where $p_g \geq \max(d+4, 2d-2)$ and $p_g - d$ is even, branched over $B \sim 6\Delta_0 + (p_g + 3d + 2)\Gamma$, or
4. ($K^2 = 4, 6,$ or 8) a double cover of the Hirzebruch surface $\mathbb{F}_{p_g - 2}$ branched over $B \sim 6\Delta_0 + (4p_g - 4)\Gamma$.

In each case, the branch curve has at most ADE singularities.

We call a surface as in Theorem 2.11 a *Horikawa surface*. These surfaces are key to the proof of Theorem 2.9.

Proof of Theorem 2.9. By taking a resolution of Du Val singularities $W' \to W$, we can assume that W has no Du Val singularities. We first show that if p is a $\frac{1}{4}(1,1)$ singularity and $K_W^2 \geq 3$, then S is of general type. Since $K_W^2 \geq 3$ and $K_W^2 = 2p_g - 3$, we have $p_g \geq 3$.

Because p has length 1, we have $K_X^2 = K_W^2 - 1 = 2p_g - 4 \geq 2$. Thus, $K_S^2 \geq K_X^2 \geq 2$. By the Enriques-Kodaira classification, S is of general type.

Now suppose that S is of general type. Then S satisfies Noether's inequality $K_S^2 \geq 2p_g - 4$. On the other hand, by Lemma 2.4, we have $K_S^2 < K_W^2 = 2p_g - 3$. Therefore $K_S^2 = 2p_g - 4$. Since the maps π and ϕ in Figure 5 do not affect the invariants p_g and q, the surface S must be a Horikawa surface. Furthermore, we have that $K_W^2 = K_S^2 - 1$, so by Theorem 2.6, the only possible Wahl singularities on W have length 1 or 2.

If $p \in W$ is a Wahl singularity of length 2, then the resolution of p in X is a T-string $\{C_1, C_2\}$ where, without loss of generality, $C_1^2 = -2$ and $C_2^2 = -5$. Since $K_X^2 = K_W^2 - 2 = K_S^2 - 1$, the surface X is the blowup of S in a single point. Let E be the exceptional curve of π. We have:

$$K_X = \phi^* K_W - \frac{1}{3}C_1 - \frac{2}{3}C_2 \tag{2.4}$$

$$K_X = \pi^* K_S + E \tag{2.5}$$

We multiply Equation (2.5) with C_1 and C_2 and use that K_S is nef to find that $E \cdot C_1 = 0$ and $E \cdot C_2 \leq 3$. On the other hand, if we multiply Equation (2.4) with E and use that K_W is nef, we see that $E \cdot C_2 \geq 2$.

If $E \cdot C_2 = 3$, then $\pi^* K_S \cdot C_2 = 0$, so $K_S \cdot \pi(C_2) = 0$. Since K_S is bif and nef, the only possibility is that $\pi(C_2)$ is a (-2)-curve. But $\pi(C_2)$ is singular, so this is not possible.

Now suppose that $E \cdot C_2 = 2$. Then $K_S \cdot \pi(C_2) = 1$ and $\pi(C_2)^2 = -1$. This implies that $\pi(C_2)$ is a nodal or cuspidal cubic. We will use the fact that S is a Horikawa surface to show that in fact such a curve cannot exist on S.

By Theorem 2.11, the surface S is the minimal resolution of a surface Y with at most Du Val singularities, which is in turn a double cover of Z where Z is either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_d$. Let $\psi : S \to Y$ be the minimal resolution of Y and $f : Y \to Z$ the double cover branched over a curve B. See Figure 6.

We must consider four cases, corresponding to the cases in Theorem 2.11. Let $C = \pi(C_2)$, and let $D = f(\psi(C))$ be the image of C on Z.

$$\begin{tikzcd}
 & X \arrow[dl, "\phi"'] \arrow[dr, "\pi"] & & & \\
W & & S \arrow[r, "\psi"] & Y \arrow[r, "f"] & Z
\end{tikzcd}$$

Figure 6. The surfaces W, X, S, Y and Z and their corresponding maps. Here, Z is either $\mathbb{P}^2$ or $\mathbb{F}_d$ for some d.

Case I. ($K^2 = 2$) Suppose that $Z = \mathbb{P}^2$ and $B \sim 8H$, where H is a hyperplane class. Then $K_Y = f^*(-3H + 4H) = f^*(H)$, so

$$\begin{aligned} 1 = K_S \cdot C &= \psi^* K_Y \cdot C & (2.6) \\ &= K_Y \cdot \psi(C) & (2.7) \\ &= f^*(H) \cdot \psi(C). & (2.8) \end{aligned}$$

Since $f^*H \cdot \psi(C)$ is odd, this implies that $f^*H \cdot \psi(C) = H \cdot D = 1$, so $D \sim H$.

But then $\psi^*(f^*(f(\psi(C))))$ is a union of smooth curves meeting transversally, one component of which is C, whereas C is singular.

Case II. ($K^2 = 8$) If $Z = \mathbb{P}^2$ and $B \sim |10H|$, then $K_Y = f^*(2H)$. In particular $K_Y \cdot F$ is even for any F. However, $K_Y \cdot \psi(C) = K_S \cdot C = 1$, so this case is impossible.

Case III. Suppose that $Z = \mathbb{F}_d$ and $B \sim |6\Delta_0 + (p_g + 3d + 2)\Gamma|$ where $p_g \geq \max(d + 4, 2d - 2)$ and $p_g - d$ is even. Then

$$K_Y = f^* \left(\Delta_0 + \frac{p_g + d - 2}{2} \Gamma \right).$$

We know $K_Y \cdot \psi(C) = 1$, so if $f(\psi(C)) \sim (a\Delta_0 + b\Gamma)$ where a and b are nonnegative, then

$$a \frac{m - d - 1}{2} + b = 1.$$

Since $p_g \geq d + 4$ and $f(\psi(C))$ is irreducible, there are two possibilities: $f(\psi(C)) \sim \Delta_0$ or $f(\psi(C)) \sim \Gamma$. But then in either case, $\psi^*(f^*(f(\psi(C))))$ is a union of smooth curves meeting transversally, with C as one of the components, whereas C is singular. Therefore, this case is impossible.

Case IV. Suppose that $Z = \mathbb{F}_{p_g-2}$ and $B \sim 6\Delta_0 + 4(p_g - 1)\Gamma$. In this case, $K_Y = f^*(\Delta_0 + (p_g - 2)\Gamma)$. If $f(\psi(C)) \sim (a\Delta_0 + b\Gamma)$, where a and b are nonnegative, then intersecting $f(\psi(C))$ with $\Delta_0 + (p_g - 2)\Gamma$ implies that $b = 1$. Since $f(C)$ is irreducible, we have that $a = 0$, and so $f(\psi(C)) \sim \Gamma$. But again, $\psi^*(f^*(f(\psi(C))))$ is a union of smooth curves meeting transversally, with C as one of the components, whereas C is singular, and we have a contradiction.

Therefore the only possible length Wahl singularity on W has length 1, so is a $\frac{1}{4}(1,1)$ singularity. $\square$

CHAPTER 3

STABLE NUMERICAL QUINTIC SURFACES WITH A UNIQUE $\frac{1}{4}(1,1)$ SINGULARITY

A stable numerical quintic surface W is a stable surface with $K^2 = 5$, $p_g = 4$ and $q = 0$. We classify all stable numerical quintic surfaces W whose unique non Du Val singularity is a $\frac{1}{4}(1,1)$ singularity. By Theorem 2.9, the minimal resolution $\phi : X \to W$ is a minimal surface such that $K_X^2 = K_W^2 - 1 = 4$, $p_g = 4$ and $q = 0$, so X is a Horikawa surface. Moreover, X contains a (-4)-curve C, the exceptional divisor of ϕ. On the other hand, given a Horikawa surface with $K^2 = p_g = 4$ and $q = 0$ and containing a (-4)-curve, we can contract C to obtain a stable numerical quintic surface with a unique $\frac{1}{4}(1,1)$ singularity. Thus, the classification of surfaces such as W becomes a question of classifying all Horikawa surfaces with $K^2 = p_g = 4$ and $q = 0$ that contain a (-4)-curve.

Theorem 2.9 suggests that in order to describe surfaces W "one above the Noether line" whose unique non Du Val singularity is a $\frac{1}{4}(1,1)$ singularity, we might instead describe pairs (X, C), where X is a Horikawa surface and C is a (-4)-curve contained in X. Because Horikawa surfaces are all described as minimal resolutions of double covers $f : Y \to Z$, we can attempt to "find" a (-4) curve on a Horikawa surface by describing how such a (-4) curve must arise from a curve on Z intersecting the branch locus in a certain way.

3.1 Double covers

Let $f: Y \to Z$ be a double cover of a smooth surface Z branched over a curve B with at most ADE singularities, and let $\psi : X \to Y$ be the minimal model of Y, obtained by resolving all Du Val singularities on Y. Then by [Hor75, Lemma 5], the surface X is the double cover of a smooth surface $\tilde{Z}$ with smooth branch locus B' obtained as follows:

Let $p = p_0$ be a singular point of $B = B_0$ and let $\sigma_1 : Z_1 \to Z = Z_0$ be the blowup of Z at p. Let E_1 be the exceptional divisor of σ_1, and let $B_1' = \sigma^*(B) - 2E_1$. Define $f_1 : Y_1 \to Z_1$ to be the double cover of Z_1 branched over B_1'. Then there exists a map $\psi_1 : Y_1 \to Z_1$ such that the following diagram is commutative.

$$\begin{tikzcd} Y_1 \arrow[r, "f_1"] \arrow[d, "\psi_1"'] & Z_1 \arrow[d, "\sigma_1"] \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

If B_1' is smooth, then Y_1 is smooth and so we can take $B' = B_1'$, $X = Y_1$, $\tilde{Z} = Z$ and $\tilde{f} = f_1$. Otherwise, repeat the process, taking p to be a singularity of B_1'. In this way, we obtain a map $\sigma : \tilde{Z} \to Z = Z_0$ which is a composition of maps $\sigma_1 \circ \cdots \circ \sigma_m$ where $\sigma_i : Z_i \to Z_{i-1}$ is the blowup of a single smooth point $p_{i-1} \in Z_{i-1}$, where p_{i-1} is singular point of $B_i' = \sigma_i^*(B_{i-1}') - 2E_i$.

We remark that the resolution given is not necessarily the log resolution of B, because we consider singularities of the curves $B_i' = \sigma_i^*(B_{i-1}') - 2E_i$, as opposed to non-nodal singularities of the preimage of B.

Now suppose that D is a smooth curve contained in Z, and let $\tilde{D}$ be the proper transform of D under the map σ. We denote by $(B \cdot D)_p$ the local intersection multiplicity of B and D at $p \in B \cap D$. If $p \in B \cap D$ is an ADE singularity of B, let D_i be the proper transform of D under $\sigma_1 \circ \cdots \circ \sigma_i$, and let q_i be the point of D_i such that $\sigma_1 \circ \cdots \circ \sigma_i(q_i) = p$. Then we can rearrange the blowups so that $q_j = p_j$ for $j \leq l$ and $q_j \neq p_j$ for $j > l$. That is, l is the smallest integer for which either B_l' is smooth at q_l or B_l' does not contain q_l.

In addition, all maps $\sigma_{l+1}, \ldots, \sigma_m$ blowup points away from $q_l \in D_l$, so that

$$(B' \cdot \tilde{D})_q = (B'_l \cdot D_l)_{q_l}.$$

We call l the *separation number* of p and note that l depends on both the singularity of B at p as well as how the branches of B at p intersect D.

For reference, we list the number of branches for each ADE singularity in the following table.

Singularity	A_n (n even), E_6, E_8	A_n (n odd), D_n (n odd), E_7	D_n (n even)
Branches	1	2	3

We state here three lemmas, the proofs of which are almost immediate, which will be useful in Theorem 3.4 below.

Lemma 3.1. Suppose that $p \in B \cap D$ is an ADE singularity of B and that D is smooth. Then $(B'_1 \cdot D_1)_{q_1} = (B \cdot D)_p - 2$. In particular, if l is the separation number of p, then $(B' \cdot \tilde{D})_q = (B \cdot D)_p - 2l$.

Proof. We have

$$(B'_1 \cdot D_1)_{q_1} = ((\sigma^* B - 2E_1) \cdot (\sigma^* D - E)) = (B \cdot D)_p - 2,$$

as desired. □

Lemma 3.2. Suppose that the branch locus B of f is reducible and contains an irreducible smooth curve D. Let $\bar{B} = B - D$ and let p be a point of $D \cap \bar{B}$. Let $\bar{B}_1 = B'_1 - D_1$. Then $(\bar{B}_1 \cdot D_1)_{q_1} = (\bar{B} \cdot D)_p - 1$. In particular, the separation number of p is equal to the local intersection $(\bar{B} \cdot D)_p$.

Proof. Since D is smooth and B has ADE singularities, any singularity of B has either 2 or 3 branches at p, of which D is locally a smooth one. If B has two branches at p, then p is either an A_n singularity of B for n odd, a D_n singularity of B for n odd, or an E_7 singularity of B. If B has 3 branches at p, then p is a D_n singularity of B for n even.

In each case, $B_1' = \sigma_1^*(\bar{B}) - E_1 + \sigma_1^*(D) - E_1$. Since

$$((\sigma_1^*\bar{B} - E_1) \cdot (\sigma_1^* D - E_1))_{q_1} = (\bar{B} \cdot D)_p - 1,$$

we have obtained the desired result. $\square$

Lemma 3.2 says in particular that if $\bar{B} \cap D$ consists of r singularities $A_{n_1}, \ldots, A_{n_r}$ of B, s singularities $D_{m_1}, \ldots, D_{m_s}$ of B with separation number 2 each, w singularities $D_{k_1}, \ldots, D_{k_w}$ of B with separation number $\frac{k_i}{2}$ each, and t E_7 singularities of B, then

$$2s + 3t + \sum_{i=1}^{r} \left(\frac{n_i + 1}{2} \right) + \sum_{i=1}^{w} \frac{k_w}{2} = (\bar{B} \cdot D)$$

and

$$\tilde{D}^2 = D^2 - \left(2s + 3t + \sum_{i=1}^{r} \left(\frac{n_i + 1}{2} \right) + \sum_{i=1}^{w} \frac{k_w}{2} \right) = D^2 - (\bar{B} \cdot D).$$

Given $g(y) = y^k(a_k + a_{k+1}y + \text{h.o.t.}) \in \mathbb{C}[[y]]$, where $a_k \in \mathbb{C}^*$, we call k the *minimal degree* of $g(y)$, and take $k = \infty$ if $g(y) = 0$.

Lemma 3.3. Suppose that $p \in B \cap D$ is an E_8 singularity of B. Then $(B \cdot D)_p$ is either 3 or 5.

Proof. Note that B is unibranched and has multiplicity 3 at p. Thus, if the tangent cone of B at p is transversal to D, then $(B \cdot D)_p = 3$. On the other hand, if the tangent cone of B at p is tangent to D, then choose coordinates on Z so that B has local equation $x^3 + y^5$. Then D is locally given by $x - f(y)$ where $f(y)$ has minimal degree $k \geq 2$.Then $(B \cdot D)_p$ is the minimal degree of $f(y)^3 + y^5$. Since $f(y)$ has minimal degree at least 2, this implies that $(B \cdot D)_p = 5$. $\square$

3.2 The classification

By Horikawa [Hor76a], there exist maps $\hat{\psi} : X \to \hat{Y}$ and $\hat{f} : \hat{Y} \to \hat{Z}$, where $\hat{Y}$ is the canonical model of X and $\hat{f} : \hat{Y} \to \hat{Z}$ is a double cover of a singular or smooth quadric, with branch locus away from the singularity of $\hat{Z}$. By resolving both A_1 singularities of $\hat{Y}$ coming from the A_1 singularity on $\hat{Z}$, we have maps $\psi : X \to Y$ and $f : Y \to Z$ where Z is either $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$, as in the diagram below.

$$W \xleftarrow{\phi} X \xrightarrow{\psi} Y \xrightarrow{f} Z$$

Let Γ be a fiber of Z and Δ an irreducible curve in the linear system $|1,1|$ on $\mathbb{F}_0$ or $|\Delta_0 + 2\Gamma|$ on $\mathbb{F}_2$. Letting B denote the branch locus of f, we have $B \sim 6\Delta$.

We describe all possible images of C under the maps ψ and f. Let $D = f(\psi(C))$ be the image of C on Z and let p be a point of $B \cap D$.

In what follows, we use the notation of Section 3.1.

Theorem 3.4. There is a one-to-one correspondence between stable numerical quintic surfaces with at most Du Val singularities and a unique $\frac{1}{4}(1,1)$ singularity, and triples (Z, B, D), where $Z = \mathbb{F}_d$ for $d = 0$ or 2, $B \sim 6\Delta$ has at most ADE singularities, and $D \sim \Gamma$ or $D \sim \Delta$ intersects B as follows:

1. $D \sim \Gamma$, there exists $p \in D \cap B$ such that $(B \cdot D)_p$ is odd, and B has either 1 or 2 singularities along D and intersects D transversally elsewhere. Moreover,

   (a) if two singularities of B are contained in D, then each singularity p has separation number 1, and either $(B \cdot D)_p = 2$ or $(B \cdot D)_p = 3$.

   (b) if one singularity p of B is contained in D, then p has separation number 2, and either $(B \cdot D)_p = 4$ or $(B \cdot D)_p = 5$.

   Figures 7, 8, and 9 show all possible ways B and D may intersect in this case.

2. $D \sim \Delta$, $D \not\subset B$, and for all $p \in D \cap B$, $(B \cdot D)_p$ is even.

3. $D \sim \Delta$ and $D \subset B$.

Proof. Suppose that W is a stable numerical quintic surface whose unique non Du Val singularity is a $\frac{1}{4}(1,1)$ and let X be its minimal resolution. Then X is a Horikawa surface with $K^2 = p_g = 4$ and $q = 0$, containing a (-4)-curve C. Let $\hat{\psi} : X \to \hat{Y}$ be the canonical model of X, so that $\hat{Y}$ has at most Du Val singularities. As discussed above, $\hat{Y}$ is a double cover of a smooth or singular quadric $\hat{Z}$, with branch locus away from any singularity of $\hat{Z}$. We resolve both A_1 singularities of $\hat{Y}$ lying over the singularity of $\hat{Z}$. Then there exists a map $\psi : X \to Y$, where Y is the double cover $f : Y \to Z$ of Z, where $Z = \mathbb{F}_2$ or $\mathbb{F}_0$, branched over $B \sim 6\Delta$ with at most ADE singularities [Hor76a]. We claim that the curve $D = \psi(f(C))$ is linearly equivalent to either Δ or Γ.

The canonical class K_Z of Z is linearly equivalent to -2Δ. Let L be a divisor such that $B \sim 2L$. Then since f is a double cover, the canonical class K_Y is given by $f^*(K_Z + L) = f^*(\Delta)$. Thus, $K_Y \cdot f^*D = 2\Delta \cdot D$.

Let $\bar{C} = \psi(C) \subset Y$. If D is not contained in the branch locus B, then $f^*(D)$ is either a union of two curves $\bar{C}$ and $\bar{C}'$ or $f^*D = \bar{C}$, depending upon how the curve D intersects the branch locus B. More precisely, $f^*(D) = \bar{C} + \bar{C}'$ if and only if the multiplicity of B and D is even at each point of intersection. We consider the three cases, $f^*(D) = \bar{C}$, $f^*(D) = \bar{C} + \bar{C}'$, and $D \subset B$, separately.

Case I. Suppose that there exists $p \in D \cap B$ such that $(B \cdot D)_p$ is odd. Then $f^*(D) = \bar{C}$ and we have

$$2\Delta \cdot D = K_Y \cdot f^*(D) = K_Y \cdot \bar{C} = 2,$$

so $\Delta \cdot D = 1$. Since C is irreducible the curve D is also irreducible. Thus, $D \sim \Gamma$. Note that $B \cdot D = 6$.

On the other hand, since $\tilde{f}$ is the double cover of a smooth surface and $C^2 = -4$, the curve $\tilde{f}(C)$ is a (-2)-curve $\tilde{D}$ on $\tilde{Z}$. Since $\tilde{D}$ has genus 0 and $\tilde{f}$ is a double cover, the Riemann–Hurwitz formula gives $B' \cdot \tilde{D} = 2$. Because C is smooth, the branch divisor B' intersects $\tilde{D}$ transversally. Commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d, "\psi"'] & \tilde{Z} \arrow[d, "\sigma"] \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

implies that $\sigma(\tilde{D}) = D$. Noting that $D^2 = 0$ and $\tilde{D}^2 = -2$ we see that the map σ blows up exactly two points p_1 and p_2 on D, which may be infinitely near.

Suppose that p_1 and p_2 are distinct, and let $p = p_1$. Then p has separation number 1. Moreover, because C is smooth, either B' intersects $\tilde{D}$ transversally at q, or B' and D do not intersect at q. That is, $(B' \cdot \tilde{D})_q = 0$ or 1. By Lemma 3.1, this implies that $(B \cdot D)_p = 2$ or 3. Conversely, if $(B \cdot D)_p = 2$ or 3, then since B is singular, p has separation number 1.

If $(B \cdot D)_p = 2$, then p is an A_n singularity of B, and any branches of B at p intersect D transversally. See Figures 7(a) and 7(b) for the local intersection of B and D.

Now suppose that $(B \cdot D)_p = 3$. If p is an A_n singularity of B for n odd, then since $(B \cdot D)_p = 3$, one branch of B intersects D transversally at p while the other intersects D at p with multiplicity 2. For $n > 1$, both branches of B are tangent to each other, so this is not possible. Thus, p is an A_1 singularity of B and B intersects D at p as in Figure 7(c).

If p is an A_n singularity for n even, then B has only one branch at p which must intersect D with multiplicity 3. This implies that the tangent cone of B at p is tangent to D. Choose local coordinates on Z so that B has local equation $x^2 - y^{n+1}$ and D has local equation $x - f(y)$ where $f(y)$ has minimal degree $k \geq 2$. Then $(B \cdot D)_p = 3$ if and only if $n = 2$. In this case, the proper transform B_1 of B is smooth and transversal to D, as desired. See Figure 7(d) for the local picture.

If B has a D_n singularity at p, where n is odd, then one branch of B is singular and the other is smooth. Since $(B \cdot D)_p = 3$, the smooth branch of B is transversal to D and the singular branch intersects D with multiplicity 2. The local intersection of B and D is shown in Figure 7(e).

If B has a D_n singularity at p, where n is even, then B has three smooth branches at p. Since $(B \cdot D)_p = 3$, each branch of B intersects D transversally at p. The local intersection of B and D is shown in Figure 7(f).

If B has an E_6 or E_8 singularity at p then B has only one branch at p, and since $(B \cdot D)_p = 3$ the tangent cone B at p is transversal to D. The local intersection of B and D are shown in Figure 7(g) and Figure 7(i).

Finally, suppose that B has an E_7 singularity at p. Then both branches of B have the same tangent cone, and since $(B \cdot D)_p = 3$, the tangent cone of each branch is transversal to D. The local intersection of B and D is shown in Figure 7(h).

Figure 7 summarizes all possible singularities of B along D that may occur if σ blows up two distinct points.

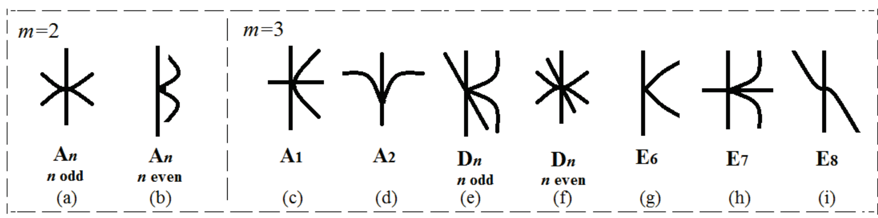


Figure 7. The possible singularities of B along D if $p_1 \neq p_2$. In each case, the vertical line represents the curve D.

We now consider the case $p_1 = p_2$. Letting $p = p_1 = p_2$, the point p has separation number 2. Moreover, because the curves $\tilde{D}$ and B' are transversal at q, we have $(B \cdot D)_p =$ 4 or 5. We show that $(B \cdot D)_p = 4$ or 5 and p has separation number 2, if and only if B and D intersect at p in one of the ways listed. In either case, by Lemma 3.1, p has separation number at most 2. Thus, we need $(B \cdot D)_p = 4$ or 5 and B'_1 singular at q_1.

Consider the case $(B \cdot D)_p = 4$. Suppose that p is an A_n singularity of B for n odd. If $n = 1$, then B_1 is smooth, so p has separation number 1. If $n > 1$, then both branches of B at p have the same tangent cone, which must be tangent to D. Choose coordinates so that the local equation of B is $x^2 - y^{n+1}$. Then the local equation of D is of the form $x - f(y)$, where the minimal degree k of $f(y)$ is at least 2. Then $(B \cdot D)_p$ is the minimal degree of the power series $[f(y)]^2 - y^{n+1}$. Thus, $(B \cdot D)_p = 4$ if and only if

(1) $n = 3$ and $k > 2$ (Figure 8(a)),

(2) $n = 3$ and $f(y) = ay^2 +$ h.o.t. for $a \neq 1$ (Figure 8(b)), or

(3) $n > 3$ and $k = 2$ (Figure 8(c)).

Note that in each case, B_1 is singular at q_1, so p has separation number 2.

If p is an A_n singularity of B for n even, then B has only one branch at p whose tangent cone is tangent to D. We can choose coordinates so that B has local equation $x^2 - y^{n+1}$ at p. Then D is locally of the form $x - f(y) = 0$, where $f(y)$ has minimal degree $k \geq 2$. Then $(B \cdot D)_p$ is the minimum degree of $[f(y)]^2 - y^{n+1}$. Since n is even, the local intersection $(B \cdot D)_p$ is 4 if and only if $k = 2$ and $n > 2$. Since $n > 2$, B_1' is singular at q_1, so p has separation number 2. Figure 8(d) shows the local intersection of B and D.

Now suppose that $(B \cdot D)_p = 4$ and p is a D_n singularity of B for n odd. Then B has two branches at p whose tangent cones are transversal to each other. Thus only one of the branches of B has tangent cone parallel to D. Suppose it is the singular branch. Then the multiplicity of the singular branch of B and D is $(B \cdot D)_p - 1 = 3$. Choose coordinates so that the singular branch of B at p has local equation $x^2 - y^{n-2}$ and the local equation of D is $x - f(y)$, where $f(y)$ has minimal degree $k \geq 2$. Since n is odd, the minimal degree of $f(y)^2 - y^{n-2}$ is either $2k$ or $n-2$, and since $(B \cdot D)_p = 4$, we have $n - 2 = 3$. Thus, $n = 5$, so p is a D_5 singularity of B. See Figure 8(e) for a visualization of how B and D intersect at p. Since B_1' is singular at q_1, p has separation number 2.

Keeping with the case $(B \cdot D)_p = 4$, suppose that p is a D_n singularity of B for odd n such that the smooth branch of B at p is tangent to D. We can choose local coordinates so that B is locally given by $x(y^2 - x^{n-2})$ and the local equation of D is of the form $x - f(y) = 0$ where $f(y)$ has minimal degree $k \geq 2$. We note that since $D \not\subset B$, we have $f(y) \neq 0$. With these coordinates, the local intersection $(B \cdot D)_p = 4$ is the minimal degree of $f(y)[y^2 - (f(y))^{n-2}]$. Since $k > 1$ and $n \geq 3$, this implies that $k = 2$. Since $n \geq 3$, the curve B_1' is singular at q_1 as desired. See Figure 8(f) for the local intersection of B and D at p.

If p is a D_n singularity of B for n even, then B has three smooth branches at p. Since

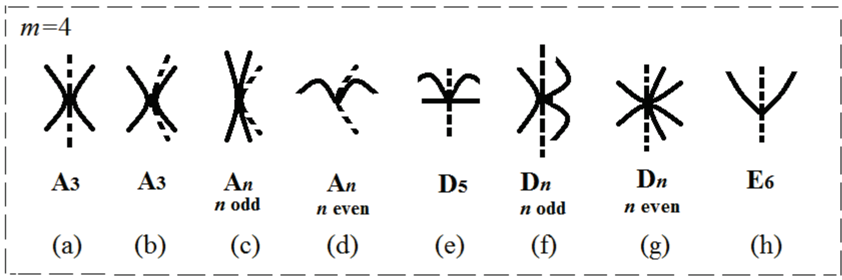


Figure 8. The possible singularities of B along D if $p_1 = p_2$ and $(B\cdot D)_p = 4$. In each case, the dashed line represents D.

$(B\cdot D)_p = 4$, two of the branches are transversal to D and the third is tangent to D with multiplicity 2. For $n \geq 6$, two branches of B have the same tangent cone, so the branch locus B intersects D at p as in Figure 8(g). The local picture for $n = 4$ is similar, except that two branches of B at p are transversal to D.

If p is an E_6 singularity of B, then B has a single branch at p whose tangent cone is tangent to D. Choose coordinates so that $x^3 - y^4$ is the local equation of B at p and the local equation of D at p is of the form $x - f(y)$, where the minimal degree of $f(y)$ is at least 2. Then $(B \cdot D)_p$ is the minimal degree of $f(y)^3 - y^4$, so $(B \cdot D)_p = 4$ as desired. The intersection of B and D at p is shown in Figure 8(h).

If p is an E_7 singularity of B, then the tangent cone of each branch of B at p is tangent to D. Choose coordinates so that B is locally given by $x(x^2 - y^3)$ and D has local equation $x - f(y)$, where $f(y)$ has minimal degree $k \geq 2$. Then the local intersection $(B \cdot D)_p$ is the minimal degree of $f(y)^3 - y^3 f(y)$. Since there is no integer k for which $3k = k+3$, the local intersection $(B\cdot D)_p$ is the minimum of $3k$ and $k+3$. But we require $(B \cdot D)_p = 4$, and since $k \geq 2$, this is impossible.

By Lemma 3.3, p is not an E_8 singularity.

See Figure 8 for a summary of the ways in which B and D intersect at p if $p_1 = p_2$ and $(B \cdot D)_p = 4$.

We move on to the case $(B\cdot D)_p = 5$. We describe all possible singularities of B along D in this case. Noting that p has separation number 2, we see that at least one branch of B at p must be tangent to D.

Suppose that p is an A_n singularity of B where n is odd. If $n = 1$, then the singularity of B at p is resolved after a single blowup. Thus, we can assume that $n > 1$. Choose coordinates so that the local equation of B near p is $x^2 - y^{n+1} - by^{n+2}$ for some $b \in \mathbb{C}$, and the local equation of D at p is $x - f(y)$ where $f(y) = 0$ or $f(y) = a_k y^k + a_{k+1} y^{k+1} +$ h.o.t. for some $k \geq 2$. Then $(B \cdot D)_p$ is the minimal degree of $f(y)^2 - y^{n+1}$. Because $(B \cdot D)_p = 5$ and $n + 1$ is even, we know that $f(y) \neq 0$. In fact, $(B \cdot D)_p = 5$ if and only if $n + 1 = 2k$, $a_k = 1$, and a_{k+1} is nonzero. Then the minimal degree of $f(y)^2 - y^{n+1}$ is $5 = 2k+1 = n+2$, so $k = 2$ and $n = 3$. The intersection of B and D at p is shown in Figure 9(a). In this case, $d_1 = 2$ and D_1 is transversal to one branch of B_1 at q_1 and tangent to the other branch with multiplicity 2.

If p is an A_n singularity of B where n is even, then the single branch of B at p intersects D with multiplicity 5. Let $x^2 - y^{n+1}$ be the local equation of B and $x - f(y)$ the local equation of D, where $f(y)$ or has minimal degree $k \geq 2$. Since n is even, we have $n + 1 \neq 2k$ for all k, so $(B \cdot D)_p$ is the minimum of $2k$ and $n + 1$. Thus $(B \cdot D)_p = 5$ if and only if $n = 4$ and $k \geq 3$. Then B_1 has an A_2 singularity at q_1 and the tangent cone of B_1 at q_1 is tangent to that of D_1 with multiplicity 2. See Figure 9(b) for the local picture.

Next, suppose that p is a D_n singularity of B where n is odd. Then B has two branches at p whose tangent cones are transversal to each other. Suppose the tangent cone of the singular branch S is tangent to D at p. Then the smooth branch is transversal to B at p. Since $(B \cdot D)_p = 5$, we have $(S \cdot D)_p = m - 1 = 4$. Using the same analysis as in previous cases, we let $x^2 y - y^{n+1}$ be the local equation of B and we see that this case occurs as long as $n \geq 5$ and Dhas local equation of the form $x - f(y)$, where $f(y)$ has minimal degree 2. See Figure 9(c) for the local picture of B and D. In this case, $d_1 = 3$ and the tangent cone of each branch of $B_1 + E_1$ is transversal to D_1 at q_1 as desired.

If p is a D_n singularity of B for n odd such that the singular branch of B at p

has tangent cone transversal to D, then the smooth branch is tangent to D at p with multiplicity 3. See Figure 9(d) for the local picture. In this case, B_1 is tangent to D_1 at q_1 and $d_2 = 2$.

If p is a D_n singularity of B where n is even, then B has three smooth branches at p. Since $(B \cdot D)_p = 5$, either two branches of B are tangent to D at p with multiplicity 2 each and the third is transversal, or two are transversal to D and the third is tangent to D with multiplicity 3. In the former case, p is a D_6 singularity and B intersects D at p as in Figure 9(e). Here, $d_1 = 2$ and B_1 is smooth at q_1 and is tangent to D_1 at q_1 with multiplicity 2, so $q \in B'$. In the latter case, n has no further restrictions and the local intersection is shown in Figure 9(f). In this case, both branches of B_1 are transversal to D_1 at q_1 and $d_1 = 3$, so $q \in B'$.

We showed above that if B has an E_6 singularity at p such that the tangent cone of B at p is tangent to D, then $(B \cdot D)_p = 4$, so this singularity does not occur if $(B \cdot D)_p = 5$.

If p is an E_7 singularity of B, then both branches of B at p are tangent to D. The singular branch intersects D at p with multiplicity at least 3, and since $D \not\subset B$, the smooth branch must be tangent D at p with multiplicity at least 2. Thus, the smooth branch of D at p is tangent to D with multiplicity 2 and the singular branch intersects D with multiplicity 3. See Figure 9(g) for the local picture. In this case both branches of B_1 at q_1 are transversal to D_1. Since $d_1 = 3$, we have $q \in B'$ as desired.

Finally, suppose that p is an E_8 singularity of B. An analysis of the local equations of B and D as above shows that as long as the tangent cone of B at p is tangent to D, we will have $(B \cdot D)_p = 5$. In this case, the proper transform B_1 of B has a cusp at q_1 with tangent cone perpendicular to D_1 at q_1. Thus, $d_1 = 3$ and $q \in B'$ as desired. See Figure 9(h) for the local picture of B and D at p.

See Figure 9 for a summary of the ways in which B and D intersect at p if $p_1 = p_2$ and $(B \cdot D)_p = 5$. This completes our discussion of Case I.

Case II. Suppose that $D \not\subset B$ and $f^*(D) = \bar{C} + \bar{C}'$. Then for each point p of $B \cap D$

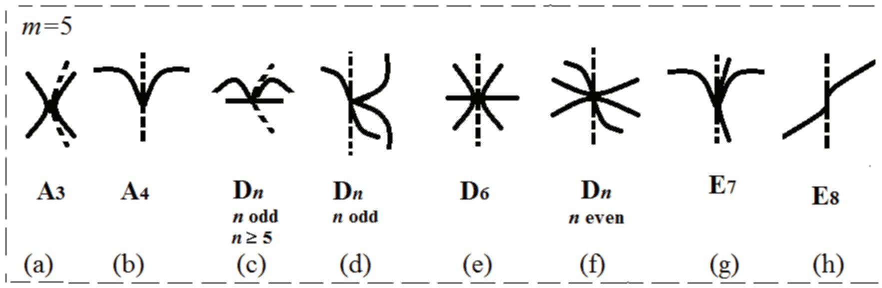


Figure 9. The possible singularities of B along D if $p_1 = p_2$ and $(B \cdot D)_p = 5$. In each case, the dashed line represents D.

the multiplicity $(B \cdot D)_p$ is even and $\bar{C}$ and $\bar{C}'$ are isomorphic and we have

$$\Delta \cdot D = \frac{1}{2} K_Y \cdot f^* D = \frac{1}{2} K_Y \cdot (\bar{C} + \bar{C}') = 2.$$

Suppose that on $\mathbb{F}_2$, we have $D \sim a\Delta_0 + b\Gamma$, where a and b are nonnegative. Then $D \cdot \Delta = b$, so that $b = 2$. Multiplying $a\Delta_0 + 2\Gamma$ by Δ_0, we see that in order for a divisor in the linear system $a\Delta_0 + 2\Gamma$ to be irreducible, we must have $a = 1$. Thus, $D \sim \Delta_0 + 2\Gamma = \Delta$. A similar calculation on $\mathbb{P}^1 \times \mathbb{P}^1$ shows that in either case $D \sim \Delta$.

We now show that if D is an irreducible curve in the linear system Δ such that at each point $p \in D \cap B$, the local intersection $(B \cdot D)_p$ is even, then $\tilde{f}^{-1}(\tilde{D})$ is a union of two (-4)-curves C and C'.

Suppose that $p_1, \ldots p_j$ are the singular points of B lying on D. Let l_i be the separation number of p_i. Then

$$\begin{aligned} (C + C')^2 &= 2\tilde{D}^2 \\ &= 2(D^2 - \sum_{i=1}^{j} l_i) \\ &= 2(2 - \sum_{i=1}^{j} l_i), \end{aligned}$$

where the second equality follows by Lemma 3.1. On the other hand

$$\begin{aligned}(C+C')^2 &= 2C^2 + 2C\cdot C' \\ &= 2C^2 + B'\cdot\tilde{D} \\ &= 2C^2 + 12 - \sum_{i=1}^{j} 2l_i \\ &= 2C^2 + 2(6 - \sum_{i=1}^{j} l_i),\end{aligned}$$

where we again use Lemma 3.1. Thus,

$$C^2 + 6 - \sum_{i=1}^{j} l_i = 2 - \sum_{i=1}^{j} l_i,$$

so $C^2 = -4$ as desired.

By Lemma 3.1, a singularity p of B along D can be either an A_n, D_n, E_6, or E_7 singularity, as long as the branches of B intersect D in such a way that the multiplicity of B and D at p is even.

Case III. If $C \subset R$ then $f^*(D) = 2C$, and so

$$2\Delta\cdot D = K_Y \cdot f^*D = K_Y \cdot 2D = 4.$$

Since D is irreducible, we must have $D \sim \Delta$. The fact that $D \subset B$ implies that $B = D + \bar{B}$ where $\bar{B}$ is in the linear system $|5\Delta|$, so $D\cdot\bar{B} = 10$. By Lemma 3.2, if p is a singularity of B contained in D, then p is either an A_n (n odd), D_n or E_7 singularity of B. Moreover, if $\bar{B}\cap D$ consists of r singularities $A_{n_1}, \ldots, A_{n_r}$ of B, s singularities $D_{m_1}, \ldots, D_{m_s}$ of B with separation number 2 each, w singularities $D_{k_1}, \ldots, D_{k_w}$ of B with separation number $\frac{k_i}{2}$ each, and t E_7 singularities of B, then

$$2s + 3t + \sum_{i=1}^{r}\left(\frac{n_i+1}{2}\right) + \sum_{i=1}^{w}\frac{k_w}{2} = (\bar{B}\cdot D) = 10.$$

Thus

$$\begin{aligned}\tilde{D}^2 &= D^2 - (\bar{B}\cdot D) \\ &= 2 - 10 \\ &= -8\end{aligned}$$

as desired.

We remark that the generic $\bar{B}$ intersects D intersect in 10 distinct points and so the double cover Y has 10 A_1 singularities. $\square$

3.3 Dimension counts

Let p be a $\frac{1}{4}(1,1)$ singularity on a stable numerical quintic surface W, let X be its minimal resolution, and let C denote the (-4) curve on X. We call W a surface of type

- 1 if $Z = \mathbb{F}_0$, $D \sim \Delta$ and B and D intersect as in Figure 10(d).
- 1' if $Z = \mathbb{F}_0$, $D \sim \Delta$ and B and D intersect as in Figure 10(e).
- 1" if $Z = \mathbb{F}_2$, $D \sim \Delta$ and B and D intersect as in Figure 10(d).
- 1"' if $Z = \mathbb{F}_0$, $D \sim \Delta$, there is a point $p \in B \cap D$ with $(B \cdot D)_p = 4$, and B and D intersect as in Figure 10(f).
- 2a if $Z = \mathbb{F}_0$, D is a fiber, and B and D intersect as in Figure 10(a).
- 2a' if $Z = \mathbb{F}_0$, D is a fiber and B and D intersect as in Figure 10(b).
- 2a" if $Z = \mathbb{F}_0$, D is a fiber, B has an A_2 singularity along D and B and D intersect as in Figure 10(c).
- 2b if $Z = \mathbb{F}_2$, D is a fiber, and B and D intersect as in Figure 10(a).

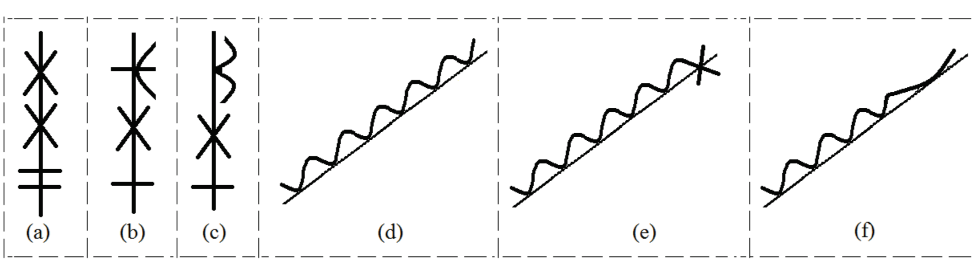


Figure 10. Six ways B and D may intersect.

Lemma 3.5. Suppose that X and X' are the minimal resolutions of stable numerical quintic surfaces W and W', each of which has a unique $\frac{1}{4}(1,1)$ singularity and no other non Du Val singularities. Let C and C' be the (-4)-curves on X and X', respectively. Let $[W]$ and $[W']$ be the points of $\overline{\mathcal{M}}_{5,5}$ corresponding to W and W', respectively. The following are equivalent:

1) $[W] = [W']$.

2) There is an isomorphism $\theta : X \to X'$ such that $\theta(C) = C'$.

3) The triples (Z, B, D) and (Z', B', D') corresponding to X and X' are isomorphic; that is, there is an isomorphism $\eta : Z \to Z'$ such that $\eta(B) = B'$ and $\eta(D) = D'$.

Proof. 1) $\iff$ 2) If $[W] = [W']$, then the surfaces W and W' are isomorphic. Since the minimal model is unique, the minimal models of W and W' are also isomorphic. Letting $\theta : X \to X'$ denote this isomorphism, it is clear that $\theta(C) = C'$. On the other hand, suppose that $\theta : X \to X'$ is an isomorphism such that $\theta(C) = C'$. Since

$$(K_X + \frac{1}{2}C) \cdot C = 0$$

and

$$(K_{X'} + \frac{1}{2}C') \cdot C' = 0,$$

the log canonical model of the pairs $(X, \frac{1}{2}C)$ and $(X', \frac{1}{2}C')$ are obtained by contracting the curves C and C', respectively. Since the log canonical model is unique and the pairs (X, C) and (X, C') are isomorphic, this implies that W is isomorphic to W'.

3) $\Rightarrow$ 2) follows by construction of X and X' from the triples given. For 2) $\Rightarrow$ 3), suppose that $\theta : X \to X'$ is an isomorphism such that $\theta(C) = C'$. Let Y and Y' be the canonical models of X and X', respectively, and denote by $\bar{C}$ and $\bar{C}'$ the images of C and C', respectively. Then the isomorphism θ induces an isomorphism of Y sending Y to Y' and $\bar{C}$ to $\bar{C}'$. The map ϕ_{K_Y} is a double cover $f : Y \to Z$, where Z is either a quadric cone or a smooth quadric. Thus, the isomorphism θ induces an isomorphism

$\eta : Z \to Z'$. Moreover, if (Z, B, D) and (Z', B', D') are the triples corresponding to X and X' under the correspondence of Theorem 3.4, then since $\theta(C) = C'$, we have $\eta(B) = B'$ and $\eta(D) = D'$, so the triples are isomorphic. $\square$

Lemma 3.6. The stable numerical quintic surfaces of types 1 and 2a correspond to 39-dimensional loci in $\overline{\mathcal{M}}_{5,5}$. Those of types 1', 1", 1"', 2a', 2a", and 2b correspond to 38-dimensional loci in $\overline{\mathcal{M}}_{5,5}$. All other types of stable numerical quintic surfaces with a unique $\frac{1}{4}(1,1)$ singularity correspond to loci of higher codimension.

Proof. Lemma 3.5 implies that each triple (Z, B, D) of Theorem 3.4 corresponds to a unique stable numerical quintic surface, up to automorphisms of Z. We count the dimension of such triples in the given cases. The main difficulty is to check that requiring that the branch divisor obtain different types of singularities at different points imposes independent conditions on B.

To create a triple (Z, B, D):

1. Fix a smooth or singular quadric Z.

2. Choose a divisor $D \sim \Delta$ or $D \sim \Gamma$. Then by Riemann-Roch, since $K_Z = -2\Delta$ and $\Delta^2 = 2$, we have $h^0(Z, \mathcal{O}(D)) = 4$ if $D \sim \Delta$ and $h^0(Z, \mathcal{O}(D)) = 2$ if $D \sim \Gamma$. Projectivizing gives a 3-dimensional space of choices if $D \sim \Delta$ and a 1-dimensional space if $D \sim \Gamma$.

3. Choose k points on D (through which B will eventually pass). Since $D \simeq \mathbb{P}^1$, we have a k-dimensional space of choices for these points. (If Z is a cone, we can choose the k points so that none of them are the singularity of Z.)

4. Choose a divisor B:

   4a. To obtain $D \not\subset B$, choose $B \sim 6\Delta$. Again by Riemann-Roch, $h^0(Z, \mathcal{O}(B)) = 49$. Projectiving gives a 48 dimensional space of possible branch curves B.

   4b. To obtain $D \subset B$, choose $B' \sim 5\Delta$. By Riemann-Roch, $h^0(Z, \mathcal{O}(B')) = 36$. Projectivizing gives a 35-dimensional space of possible branch curves B'. By abuse of notation, take $B = B'$ (and note that the resulting triple will be of the form $(Z, B' + D, D)$, or with our abuse of notation, $(Z, B + D, D)$).

5. Consider the restriction exact sequence

$$0 \to \mathcal{O}_Z(B - D) \to \mathcal{O}_Z(B) \to \mathcal{O}_D(B) \to 0.$$

By Kodaira vanishing, $H^1(Z, \mathcal{O}_Z(B - D)) = 0$. Thus, the map

$$H^0(Z, \mathcal{O}_Z(B)) \to H^0(D, \mathcal{O}_D(B))$$

is surjective, and so we can find a curve $B \in |6\Delta|$ (or $B' \in |5\Delta|$) such that the restriction of B to D passes through any m points on D, counted with multiplicities, where $m = (B \cdot D)$. Thus, the requirement that B pass through the given m points, counted with multiplicities, is a codimension m condition.

6. The group of automorphisms of Z is 6-dimensional if Z is smooth and 7-dimensional if Z is a cone. Thus, modding out by automorphisms of Z is either a codimension 6 condition or codimension 7 condition.

Triples (Z, B, D) where $D \subset B$ give a locus of dimension at most $3+10+35-10-6 = 32$, so we can assume for the rest of the proof that $D \not\subset B$.

7. There is at the moment no guarantee that the most general B is smooth at any given point, nor is it immediate that imposing the condition that B obtain a certain mild singularity at a given point does not impose conditions on B at the other $k - 1$ points. Provided the multiplicity at each point is small enough, the fact that these conditions *are* linearly independent follows from the fact that B is sufficiently big. That is, for $n \leq 5$, the divisor $B - nD$ is big and nef, so the cohomology group $H^1(Z, \mathcal{O}_Z(B - nD))$ is zero by Kodaira vanishing. Thus, the map

$$H^0(Z, \mathcal{O}_Z(B)) \to H^0(D, \mathcal{O}_{nD}(B))$$

induced by the restriction $\mathcal{O}_Z(B) \to \mathcal{O}_{nD}(B)$ is surjective. This means that we can choose B in such a way that we can require the degree $1, 2, \ldots, n-1$ parts of the "Taylor expansion" of its equation

$$s|_{nD} = s_0 + s_1 d + s_2 d^2 + ...$$

to be of any form we desire, where $d \in H^0(Z, \mathcal{O}_Z(D))$ is the equation of D and $s_i \in H^0(D, \mathcal{O}_D(B - iD))$.

Suppose we want to impose the condition that B acquires a node at a given point p for which $(B \cdot D)_p = 2$. This is equivalent to requiring that the linear term in its Taylor expansion vanish at p, and that the discriminant of the quadratic term be non-vanishing at p. Therefore, this condition has expected codimension 1. Since this is a requirement on the degree 1 and 2 parts of the Taylor expansion, taking $n = 3$ implies that the requirements that B be either smooth or obtain at most a node at each of its points are linearly independent conditions. That is, the condition that B acquire a node at a point with multiplicity 2 is indeed a codimension 1 condition.

Similarly, the requirement that B acquire an A_2 singularity at a point p for which $(B \cdot D)_p = 2$ is equivalent to requiring that the linear term in its Taylor expansion vanish at p, the discriminant of the quadratic term also vanish at p, and the cubic term be nonvanishing. Since this is a requirement on the part of the Taylor expansion of degrees 1, 2, and 3, taking $n = 4$ implies that the requirement that B aquire an A_2 singularity at the desired point is a codimension 2 condition.

The requirement that B acquire a node at a point p for which $(B \cdot D)_p = 3$ is equivalent to requiring the linear term in its Taylor expansion to vanish at p, and the coefficient of one monomial in the quadratic term to vanish at p. Again, this is a requirement on the degree 2 part of the Taylor expansion, so taking $n = 3$ implies that this is a codimension 2 condition that does not impose conditions on the other points of $B \cap D$.

Let l be the dimension of the set of triples such that $|B \cap D| = k$ (set theoretically).

Then

$$
l = \begin{cases} 33+k & \text{if } D \in |\Delta| \text{ on } \mathbb{F}_0 \\ 32+k & \text{if } D \in |\Delta| \text{ on } \mathbb{F}_2 \\ 37+k & \text{if } D \in |\Gamma| \text{ on } \mathbb{F}_0 \\ 36+k & \text{if } D \in |\Gamma| \text{ on } \mathbb{F}_2 \end{cases}.
$$

Thus, if m is the codimension of the set of triples such that B has prescribed singularities, then in order for the set of such triples to have dimension 38 or 39, we have

$$
\begin{aligned}
m = k-6 \text{ or } m = k-5 \quad & \text{if } D \in |\Delta| \text{ on } \mathbb{F}_0 \\
m = k-6 \quad & \text{if } D \in |\Delta| \text{ on } \mathbb{F}_2 \\
m = k-1 \text{ or } m = k-2 \quad & \text{if } D \in |\Gamma| \text{ on } \mathbb{F}_0 \\
m = k-2 \text{ or } m = k-3 \quad & \text{if } D \in |\Gamma| \text{ on } \mathbb{F}_2
\end{aligned}
$$

.

In particular, we see that if $D \sim \Delta$, then since $k \leq 6$, we have $m = 0$ or 1. If $D \sim \Gamma$, then since $k \leq 4$, we have $m \leq 3$.

For instance, the dimension of the locus of type 1 surfaces is $3+6+48-12-6 = 39$, and of type 1' surfaces is $3+6+48-12-1-6 = 38$.

The dimension of the locus of type 2a surfaces is $1+4+48-6-1-1-6 = 39$, and of type 2b surfaces is $1+4+48-6-1-1-7 = 38$.

Working through each of the remaining possibilities in Theorem 3.4 gives the desired result. □

The proof of the following is incomplete, although the main ingredients are Theorem 3.4 and Lemma 3.6, especially the proof of the latter.

Theorem 3.7. Let W be a stable numerical quintic surface corresponding to the pair (Z, B, D), and let $[W]$ denote its corresponding point in $\overline{\mathcal{M}}_{5,5}$. If $D \sim \Gamma$, then $[W]$ is in the closure of the locus of 2a surfaces. If $D \sim \Delta$, then $[W]$ is in the closure of the locus of surfaces of type 1. Thus, the closures of the loci of surfaces of types 1 and 2a contain all surfaces whose unique non Du Val singularity is a $\frac{1}{4}(1,1)$ singularity.

CHAPTER 4

DEFORMATIONS OF SURFACES OF TYPES 1 AND 2a

In this chapter, we describe the components of $\overline{\mathcal{M}}_{5,5}$ corresponding to surfaces of types 1 and 2a and show that their closures are generically Cartier divisors in the boundary of the type I and IIa components of $\mathcal{M}_{5,5}$. In Chapter 4, Lemma 3.6, we showed that these components are both 39-dimensional. In Section 4.1, we show that these components are in the boundary of the respective components on $\mathcal{M}_{5,5}$ by constructing explicit $\mathbb{Q}$-Gorenstein families of numerical quintic surfaces whose stable limits are of the desired form. In Section 4.3, we prove that the components $\bar{1}$ and $\overline{2\mathrm{a}}$ are generically Cartier divisors by showing that there are no obstructions to $\mathbb{Q}$-Gorenstein deformations of surfaces of types 1 and 2a.

4.1 Families of stable quintic surfaces

We use the characterization of surfaces in Theorem 3.4 to construct families of numerical quintic surfaces degenerating to a stable numerical quintic surface whose unique non Du Val singularity is a $\frac{1}{4}(1,1)$ singularity.

4.1.1 Type 1

We describe a family of quintic surfaces degenerating to a stable numerical quintic surface of type 1. The fact that the stable limit of the family is a stable numerical quintic surface of type 1 is not obvious, so one might wonder why we even considered the given family in the first place. The equation of the family was in fact suggested by a computation during a summer REU involving the Craighero-Gattazzo surface.

A numerical Godeaux surface is a minimal surface of general type with $p_g = q = 0$ and $K^2 = 1$. Examples of these surfaces include Godeaux surfaces, Barlow surfaces, and the Craighero-Gattazzo surface. The Craighero-Gattazzo surface is the minimal resolution of a quintic surface with 4 simple elliptic singularities of type $z^2 + x^3 + y^6 = 0$.

The moduli space of numerical Godeaux surfaces with trivial $H_1(S, \mathbb{Z})$ is conjecturally 8 dimensional. It is unknown whether or not this moduli space is connected. For instance, the Craighero-Gattazzo surface has ample canonical class, whereas Barlow surfaces do not. Catanese and LeBrun [CL97] proved that Barlow surfaces deform to surfaces with ample canonical class, but it remains unknown whether or not Barlow surfaces deform to the Craighero-Gattazzo surface.

Motivated by this question, Charles Boyd computed a 7-adic model of the Craighero-Gattazzo surface, which showed that the Craighero-Gattazzo surface acquires a $\frac{1}{4}(1,1)$ singularity in characteristic 7. Its form suggested the equation of the family in the following theorem.

Theorem 4.1. Consider the family $(\mathscr{X}, \Delta)$ of surfaces

$$S_t = \{q^2 l + tqf_3 + t^2 f_5 = 0\} \subset \Delta_t \times \mathbb{P}^3_{x_0,x_1,x_2,x_3}$$

where f_3 and f_5 are general forms of degrees 3, and 5, respectively and such that the surface S_t is a smooth quintic surface for $t \in \Delta^*$. Suppose that the special fiber S_0 is the union of a double quadric Q given by $q = 0$ and a plane L given by $l = 0$ intersecting transversally. Then the KSBA stable limit of the family $(\mathscr{X}, \Delta)$ is a stable numerical quintic surface of type 1. Moreover, the general stable numerical quintic surface of type 1 is the stable limit of such a family.

Proof. The singular locus of $\mathscr{X}$ is the surface Q, so $\mathscr{X}$ is not normal. To compute the stable limit we first normalize the family. After normalization and an extremal contraction, we will see that the family of surfaces obtained has reduced special fiber and ample canonical class.

Let $\nu : \mathscr{X}^{\nu} \to \mathscr{X}$ be the normalization of $\mathscr{X}$. We determine the structure of $\mathscr{X}^{\nu}$. First note that the normalization is an isomorphism away from Q.

Let U be a complex analytic neighborhood in $\mathscr{X}$ of a point $p \in Q$. Then on U, we can write

$$q|_U = q_1 + q_2, \quad l|_U = l_0 + l_1, \quad f_3|_U = \sum_{i=0}^{3} f_{3,i}, \quad f_5|_U = \sum_{i=0}^{5} f_{5,i}$$

where the subscripts indicate the degree of each term. Giving t weight 1, we can write the equation of $\mathscr{X} \cap U$ as

$$q_1^2 l_0 + t q_1 f_{3,0} + t^2 f_{5,0} + \text{higher order terms.}$$

Let $D \subset Q$ be the "discriminant curve" given by $\{f_3^2 - 4 l f_5 = 0\} \subset Q \cap U$. If $p \notin D$, then the equation of $\mathscr{X} \cap U$ factors into the product of two linear terms which are not equal. That is, $(p \in \mathscr{X})$ is locally analytically isomorphic to a threefold $\mathscr{Y} = (xy = 0) \subset \mathbb{A}^4$. Thus, over the open set $Q \backslash D \subset Q$, the special fiber $\mathscr{X}_0^{\nu}$ is an unramified double cover of $Q \backslash D$.

Now consider a point $p \in D$ and let U be a complex analytic neighborhood of $p \in \mathscr{X}$. Since $D(p) = 0$, the equation of $\mathscr{X} \cap U$ may be written locally analytically as

$$g = (q + \frac{1}{2} f_{3,0} t)^2 + \text{h.o.t.}$$

if $p \notin L$ and

$$g = t^2 + \text{h.o.t.}$$

if $p \in L$. Thus, in order to determine the structure of $\mathscr{X}^{\nu}$ near p, we must consider the degree three part of g. This is:

$$g_3 = q_1^2 l_1 + 2 q_1 q_2 l_0 + t q_1 f_{3,1} + t q_2 f_{3,0} + t^2 f_{5,1}.$$

If $p \notin L$, then we assume that $l_0 = 1$ and complete the square in the first few terms of g:

$$\begin{aligned} g &= (q_1 + \frac{1}{2}tf_{3,0})^2 + 2q_2(q_1 + \frac{1}{2}tf_{3,0}) + q_1^2 l_1 + tq_1 f_{3,1} + t^2 f_{5,1} + \text{h.o.t.} \\ &= (q_1 + \frac{1}{2}tf_{3,0} + q_2)^2 + q_1^2 l_1 + tq_1 f_{3,1} + t^2 f_{5,1} + \text{h.o.t.} \end{aligned}$$

Let $y = q_1 + \frac{1}{2}tf_{3,0}$ and note that y is a linear form. This last equation now becomes

$$g = (y + q_2)^2 + y^2\alpha + yt\beta + t^2\gamma + \text{h.o.t.}$$

where

$$\alpha = l_1,$$

$$\beta = f_{3,1} - l_1 f_{3,0},$$

and

$$\gamma = f_{5,1} - \frac{1}{2}f_{3,0}(f_{3,1} + \frac{1}{2}l_1)$$

are linear forms. Finally we can rewrite this as

$$\begin{aligned} g &= (y + q_2)^2 + (y + q_2)(y\alpha + t\beta) - q_2(y\alpha + t\beta) + t^2\gamma + \text{h.o.t.} \\ &= [(y + q_2) + \frac{1}{2}(y\alpha + t\beta)]^2 + t^2\gamma + \text{h.o.t.} \\ &= z^2 + t^2\gamma + \text{h.o.t.} \end{aligned}$$

where z is a linear form. Thus, in a complex analytic neighborhood of any point $p \in Q \cap D \backslash L$, the threefold $\mathscr{X}$ is locally analytically isomorphic to the threefold $\mathscr{Y} = \{z^2 - t^2\gamma = 0\} \subset \mathbb{A}^4_{\gamma,t,z,s}$ which is the product of $\mathbb{A}^1$ with the Whitney umbrella, or pinch point. The normalization of $\mathscr{Y}$ is $\mathbb{A}^3_{u,v,w}$ with normalization map $(u, v, w) \mapsto (u^2, v, uv, w)$.

The quadric Q corresponds to the locus $(z = t = 0) \subset \mathscr{Y}$, so the normalization $\mathscr{X}_0^\nu$ of $\mathscr{X}_0$ is the double cover of the smooth quadric Q, ramified along the discriminant curve D. On the other hand, if Q is singular and D does not intersect the singularity of Q, then $\mathscr{X}_0^\nu$ is the double cover of a singular quadric. Resolving the singularity we see that $\tilde{X}_0$ is in fact a double cover of $\mathbb{F}_2$. Since the surfaces D_0 and Q intersect in a curve of degree 12, the surface $\mathscr{X}_0^\nu$ is the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$, ramified along a divisor in the linear system $|6\Delta|$.

To determine what happens to the plane L under the normalization, we begin by assuming that $p \in L \cap Q \backslash D$. Then $l_0 = 0$ and $f_{3,0} \neq 0$, so we can assume $f_{3,0} = 1$ and we have

$$g = tq_1 + t^2 f_{5,0} + q_1^2 l_1 + tq_1 f_{3,1} + tq_2 + t^2 f_{5,1} + \text{ h.o.t..}$$

By choosing f_5 sufficiently general, we can assume that $f_{5,0} \neq 0$ and so take $f_{5,0} = 1$. Thus, g factors as

$$\begin{aligned} g &= tq_1 + t^2 + q_1^2 l_1 + tq_1 f_{3,1} + tq_2 + t^2 f_{5,1} + \text{ h.o.t.} \\ &= (t + q_1 l_1 + \text{ h.o.t.}) \cdot (t + q_1 - q_1 l_1 + \text{ h.o.t.}) \end{aligned}$$

The linear term of each factor is unique up to multiplication by a nonzero constant. In particular, we see that because the second factor contains the linear term q_1 which does not involved t or l_1, the second factor does not vanish identically along L. Since $g(p) = 0$ the second term must vanish along L. Thus, the normalization of $(p \in \mathscr{X})$ is an unramified double cover of $Q \backslash D$, of which one component (the component corresponding to the first factor of g above) contains the entire proper transform of $L \backslash D$.

For the six points $p \in L \cap Q \cap D$, we have $l_0 = 0$ and $f_{3,0} = 0$. By choosing f_5 sufficiently general, we can assume that $f_{5,0} = 1$ and so we can write the local equation of $\mathscr{X}$ as

$$g = t^2 + tq_1 f_{3,1} + q_1^2 l_1 + t^2 f_{5,1} + \text{ h.o.t.}$$

Completing the square gives

$$g = (t + \frac{1}{2} q_1 f_{3,1})^2 + q_1^2 l_1 + t^2 f_{5,1} + \text{ h.o.t.}$$

Let $\alpha = t + \frac{1}{2} q_1 f_{3,1}$ and note that we can write $t = \alpha - \frac{1}{2} q_1 f_{3,1}$. Then g can be rewritten in terms of α as

$$\begin{aligned} g &= \alpha^2 + q_1^2 l_1 + (\alpha - \frac{1}{2} q_1 f_{3,1})^2 f_{5,1} + \text{ h.o.t.} \\ &= \alpha^2(1 + f_{5,1}) + q_1^2 l_1 + \text{ h.o.t.} \\ &= y^2 + q_1^2 l_1 + \text{ h.o.t.} \end{aligned}$$

Thus, the threefold $\mathscr{X}$ is again locally analytically isomorphic to the threefold $\mathscr{Y} = \{y^2 - x^2 z = 0\} \subset \mathbb{A}^4_{x,y,z,s}$ which is the product of $\mathbb{A}^1$ with the Whitney umbrella. The normalization of $\mathscr{Y}$ is $\mathbb{A}^3_{u,v,w}$ with normalization map $(u, v, w) \mapsto (u, uv, v^2, w)$. In the coordinates of $\mathbb{A}^4_{x,y,z,s}$ the plane L corresponds to the plane $P = (z = y = 0) \subset \mathscr{Y}$. Because the normalization is an isomorphism over this locus, we have P^ν is the plane given by $v = 0$. The surface Q corresponds to the locus $(x = y = 0) \subset \mathscr{Y}$, which under the normalization becomes the plane $u = 0$. Thus, we see that the proper transforms L^ν and Q^ν of L and Q intersect transversally after the normalization.

The plane L intersects the quadric Q in a conic. Thus, for general q, l and D, the curve $L \cap Q$ intersects the locus $D \cap Q$ tangentially at 6 points. Taking the double cover of Q branched over D gives a smooth surface $\tilde{W}$ with a smooth (-4)-curve C given by the intersection of the plane L with the surface $\mathscr{X}_0^\nu$.

We now show that an extremal contraction of L results in a family of surfaces with ample canonical class. The canonical class K_{X_0} is given by $K_{\mathscr{X}^\nu}|_{X_0}$. Since $K_{\mathscr{X}^\nu}|_{\tilde{W}} = K_{\tilde{W}} + C$ and

$$K_{\mathscr{X}^\nu}|_L = K_L + C \sim -2H + H \sim -H,$$

we see that $L \subset \mathscr{X}^\nu$ can be contracted and that the surface W obtained after contracting $C \subset \tilde{W}$ gives the stable limit. Note moreover that C is a (-4)-curve on $\tilde{W}$, so this contraction produces a $\frac{1}{4}(1,1)$ singularity on W. Thus, the stable limit of the family is a stable numerical quintic surface W with a $\frac{1}{4}(1,1)$ singularity of type 1.

We claim that any stable numerical quintic surface of type 1 may be obtained as the stable limit of such a family. By Lemma 3.5, it suffices to show that given any triple

(Z, B, D) – where Z is a fixed smooth quadric and $B \sim 6\Delta$ and $D \sim \Delta$ are smooth, such that B intersects D with multiplicity 2 at 6 points – we can find a family of the desired form whose stable limit is a stable numerical quintic surface W corresponding to (Z, B, D) under the correspondence of Theorem 3.4.

Fix such a triple. Then Z is isomorphic to a smooth quadric in $\mathbb{P}^3$ given by $q = 0$. Let l be the equation of the hyperplane L in $\mathbb{P}^3$ such that $L \cap Z = D$. We claim that B is also given by $V \cap Z$, where V is a hypersurface of degree 6 in $\mathbb{P}^3$. To see this, let H be a general hyperplane section of $\mathbb{P}^3$ and consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-Z + 6H) \to \mathcal{O}_{\mathbb{P}^3}(6H) \to \mathcal{O}_Z(6H) \to 0.$$

Since $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-Z + 6H)) = H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4H)) = 0$, we see that global sections of $\mathcal{O}_{\mathbb{P}^3}(6H)$ surject onto global sections of $\mathcal{O}_Z(6H)$. Noting that $\mathcal{O}_Z(6H) \simeq \mathcal{O}_Z(6\Delta)$, this implies that the element $B \in |6\Delta|$ can be lifted to a hypersurface V of degree 6 in $\mathbb{P}^3$, proving the claim.

Next consider the exact sequence

$$0 \to \mathcal{O}_Z(V - L) \to \mathcal{O}_Z(V) \to \mathcal{O}_{Z \cap L}(V) \to 0.$$

Since B intersects D at 6 points with multiplicity 2 each, this implies that the equation of $V|_L$ is of the form f_3^2, where the six points of $B \cap D$ are given by $f_3 = q = 0$. Therefore V can be chosen to have equation $f_3^2 - l f_5$, where f_5 is a general form of degree 5. Then taking

$$S_t = \{q^2 l + tq f_3 + t^2 f_5 = 0\} \subset \Delta_t \times \mathbb{P}^3_{x_0, x_1, x_2, x_3}$$

gives the desired family. $\square$

4.1.2 Types 2a and 2b

Friedman [Fri83] constructed a family of stable numerical quintic surfaces with general fiber a numerical quintic surface of type IIb and special fiber a stable numerical quintic surface of type 2b. His construction easily generalizes to give a family of stable numerical quintic surfaces whose general fiber is a numerical quintic surface of type IIa and with special fiber a stable numerical quintic surface of type 2a.

Before continuing with the construction of the family, we describe all $\mathbb{Q}$-Gorenstein deformations of $\frac{1}{4}(1,1)$ singularities. This will enable us to see that Friedman's family induces a *versal* local $\mathbb{Q}$-Gorenstein deformation of the $\frac{1}{4}(1,1)$ singularity on the special fiber.

Let $(p \in W)$ be a germ of a $\frac{1}{4}(1,1)$ singularity. Then $(p \in W)$ is analytically isomorphic to the singularity

$$(xy = z^2) \subset \frac{1}{2}(1,1,1).$$

Any deformation of $(p \in X)$ is analytically isomorphic to a deformation of the form

$$(xy = z^2 + t^\alpha) \subset \frac{1}{2}(1,1,1) \times \mathbb{A}^1_t,$$

for some integer $\alpha > 0$ called the *axial multiplicity* of the deformation. The resolution of the total space of such a deformation consists of two components intersecting with multiplicity α. A versal local $\mathbb{Q}$-Gorenstein deformation of $(p \in X)$ has axial multiplicity 1; that is, its resolution consists of two components meeting transversally.

Theorem 4.2. [Fri83] There is a $\mathbb{Q}$-Gorenstein deformation $\mathscr{X} \to T$ where T is the unit disk in $\mathbb{C}$ such that

1. X_t is a smooth numerical quintic surface of type IIa (respectively, IIb) for $t \neq 0$; and

2. X_0 is a stable numerical quintic surface with a $\frac{1}{4}(1,1)$ singularity of type 2a (respectively, 2b).

Furthermore, this deformation induces a versal local $\mathbb{Q}$-Gorenstein deformation of a $\frac{1}{4}(1,1)$ singularity.

Proof. We follow Friedman's construction, with slight modifications in order to construct both deformations simultaneously. Let $\mathbb{F}_d$ for $d \geq 0$ be the Hirzebruch surface with 0-section Δ_0 and generic fiber Γ, and note that $\mathbb{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

We begin by recalling Horikawa's construction of numerical quintic surfaces of types IIa and IIb. Let Z be the surface $\mathbb{F}_0$ or $\mathbb{F}_2$ and let D be a fiber of Z. Let x and y be distinct points on D which do not lie on Δ_0 and let $\sigma : \tilde{Z} = \mathrm{Bl}_{x,y} Z \to Z$ be the blowup of Z in x and y. Let E_x and E_y be the exceptional divisors. Denote by $\tilde{D}$ the proper transform of D. Note that

$$\tilde{D} \sim \sigma^*(D) - E_x - E_y$$

and $\tilde{D}^2 = -2$. By abuse of notation, we let Δ_0 and Γ denote the proper transforms of the respective divisors on Z and let $\Delta \sim \Delta_0 + d\Gamma$ be an irreducible curve on $\tilde{Z} = \tilde{\mathbb{F}}_d$.

Let $B = 6\Delta + 2\Gamma - 4E_x - 4E_y$ and note that $B \cdot \tilde{D} = -2$. One can then write $B \sim B_1 + \tilde{D}$ where $|B_1|$ is basepoint free, B_1 is smooth, and $B_1 \cap \tilde{D} = \emptyset$. Thus the double cover $f : \tilde{X} \to \tilde{Z}$ branched over $B_1 + \tilde{D}$ is smooth. Note that the preimage $f^{-1}(\tilde{D})$ is $2C$ where C is a (-1)-curve. Contracting C gives a minimal numerical quintic surface X of type IIa if $Z = \mathbb{P}^1 \times \mathbb{P}^1$ or of type IIb if $Z = \mathbb{F}_2$.

We now degenerate the branch locus B by splitting off another copy of $\tilde{D}$. That is, take $B \sim B_2 + 2\tilde{D}$ where $B_2 \sim 6\Delta - 2E_x - 2E_y$. Then the linear system $|B_2|$ is basepoint free and we can choose B_2 to be smooth. We note that $B_2 \cdot \tilde{D} = 2$.

The double cover Y of $\tilde{Z}$ branched over $B_2 + 2\tilde{D}$ is the same as the double cover of $\tilde{Z}$ branched over B_2 and is, by Theorem 3.4, the minimal resolution of a stable numerical quintic surface of type 2a if $Z = \mathbb{F}_0$ or of type 2b if $Z = \mathbb{F}_2$.

The explicit construction of this family and its semistable model may be found in [Fri83]. Following Friedman's construction, we obtain a family

$$\pi : \tilde{\mathscr{X}} \to T$$

whose generic fiber is a numerical quintic surface of type IIa if $Z = \mathbb{F}_0$ (respectively, IIb if $Z = \mathbb{F}_2$). The special fiber $\tilde{\mathscr{X}}_0$ is a union of surfaces $V \cup W$ intersecting transversally along a curve R, where V is the minimal resolution of a stable numerical quintic surface of type 2a (respectively, 2b) and $W \simeq \mathbb{P}^2$. Moreover, the curve $R|_V$ is a (-4)-curve and $R|_{\mathbb{P}^2}$ is a conic.

By adjunction, we have

$$K_{\tilde{\mathscr{X}}} \cdot R = (K_V + R) \cdot R = -2,$$

so the family $\pi : \tilde{\mathscr{X}} \to T$ is not stable. Contracting the $\mathbb{P}^2$, we obtain a $\mathbb{Q}$-Gorenstein family $\mathscr{X} \to T$ whose special fiber is a stable numerical quintic surface with a unique $\frac{1}{4}(1,1)$ singularity of type 2a (respectively, 2b) if $Z = \mathbb{F}_0$ (respectively, $Z = \mathbb{F}_2$).

We note that the $\mathbb{Q}$-Gorenstein deformation $\mathscr{X} \to T$ induces a versal local $\mathbb{Q}$-Gorenstein deformation of a $\frac{1}{4}(1,1)$ singularity, because the special fiber of the family $\tilde{\mathscr{X}} \to T$ consists of two components meeting transversally. $\square$

Remark 4.3. In [Fri83, Corollary 1.2], Friedman uses Horikawa's description of the moduli space $\mathcal{M}_{5,5}$ to deduce the existence of a $\mathbb{Q}$-Gorenstein family $\tilde{\mathscr{X}} \to T$ of smooth quintic surfaces whose special fiber is an "accordion" of surfaces $V \cup W_1 \cup W_2 \cup \cdots \cup W_n$ where V is the minimal resolution of a stable quintic surface of type 2b, $W_1, \ldots, W_{n-1}$ are copies of $\mathbb{F}_4$, and W_n is a copy of $\mathbb{P}^2$, intersecting transversally as in Figure 11.

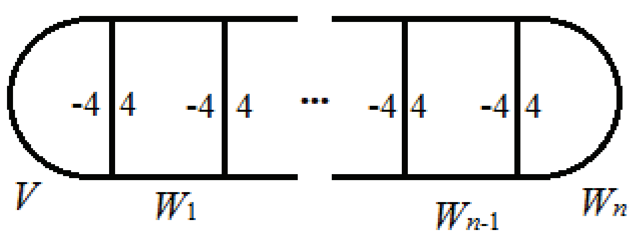


Figure 11. The special fiber of Friedman's $\mathbb{Q}$-Gorenstein family of smooth quintic surfaces. The surface V is the minimal resolution of a 2b surface, $W_1, \ldots, W_{n-1}$ are copies of $\mathbb{F}_4$, and W_n is a copy of $\mathbb{P}^2$.

Again, the canonical class $K_{\tilde{\mathscr{X}}}$ is not ample, and the stable limit of $\tilde{\mathscr{X}} \to T$ is obtained by contracting the surfaces $W_1, ..., W_n$. We now recognize the resulting special fiber as a stable numerical quintic surface of type 2b. Thus, Friedman's family is a $\mathbb{Q}$-Gorenstein smoothing of a 2b surface to a quintic surface. This family gives a local deformation of the $\frac{1}{4}(1,1)$ singularity with axial multiplicity n, so unless $n=1$, the induced deformation is not versal. In Section 5.4, we show that given a 2b surface X, there exists a $\mathbb{Q}$-Gorenstein smoothing of X to a quintic surface with $n=1$.

Friedman also raises the question of describing deformations of 2b surfaces explicitly. Theorem 5.1 answers this question.

4.2 Some sheaf calculations

Let X be a smooth surface and $D=\sum_{i=1}^{k} D_i$ a divisor in X with simple normal crossings (in particular, each component divisor D_i is smooth). Let $\Omega^1_X(\log D)$ denote the sheaf of logarithmic differentials. There is a short exact sequence of sheaves

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \bigoplus_{i=1}^{k} \mathcal{O}_{D_i} \to 0$$

where the map $\Omega^1_X(\log D) \to \bigoplus_{i=1}^{k} \mathcal{O}_{D_i}$ is the residue map.

Now let W be a surface whose only non Du Val singularity is a Wahl singularity and let X be its minimal resolution. If D is the exceptional divisor on X, then one can show that obstructions to $\mathbb{Q}$-Gorenstein deformations of W lie in the cohomology group $H^2(X, T_X(-\log D))$ [LP07]. Thus, if the minimal resolution X of a stable numerical quintic surface with exceptional (-4)-curve C satisfies $H^2(X, T_X(-\log C))=0$, then the locus of such surfaces is generically smooth in $\overline{\mathcal{M}}_{5,5}$.

The calculation of $H^2(X, T_X(-\log C))$ in Sections 4.3 and 5 requires the following lemmas.

Lemma 4.4. Let $\sigma : Y \to Z$ be the blowup of a smooth surface at a point p lying in the smooth locus of a divisor $D \subset Z$ with normal crossings. Let $\tilde{D} \subset Y$ be the proper transform of D. Then $\sigma_*\Omega^1_Y(\log \tilde{D}) = \Omega^1_Z(\log D) \otimes \mathfrak{M}_p$, where $\mathfrak{M}_p$ is the ideal sheaf of p on Z.

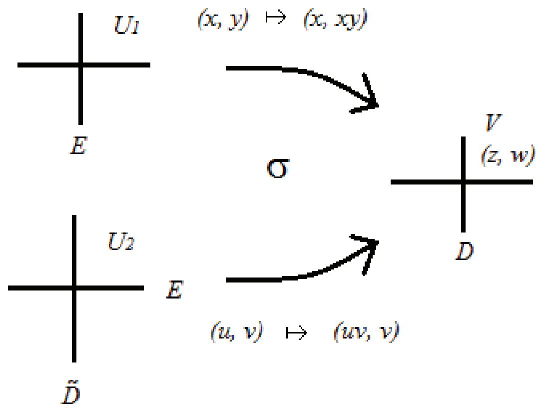


Figure 12. The map σ.

Proof. It suffices to show the equality in a neighborhood of the exceptional divisor E. Let $V \subset Z$ be a coordinate neighborhood around p. Choose coordinates (z, w) on V so that p is at the origin and the local equation of D is z. Then $\sigma^{-1}(V)$ is covered by two neighborhoods U_1 and U_2. Choose coordinates (x, y) on U_1 so that $\sigma(x, y) = (x, xy)$ and the local equation of $E \cap U_1$ is x. Note that $\tilde{D}$ does not appear in U_1. Let coordinates on U_2 be (u, v) so that $\sigma(u, v) = (uv, v)$. On U_2, the local equation of E is v and the local equation of $\tilde{D}$ is u. See Figure 12.

On U_1, we have

$$\Omega^1_Y(\log \tilde{D})(U_1) = \left\{ f\left(z, \frac{w}{z}\right) dz + g\left(z, \frac{w}{z}\right) d\left(\frac{w}{z}\right) \,\middle|\, f, g \in \mathcal{O}_Z(V) \right\} \tag{4.1}$$

$$= \left\{ \left[f\left(z, \frac{w}{z}\right) - \frac{w}{z^2} g\left(z, \frac{w}{z}\right) \right] dz + \frac{1}{z} g\left(z, \frac{w}{z}\right) dw \,\middle|\, f, g \in \mathcal{O}_Z(V) \right\} \tag{4.2}$$

On U_2

$$\Omega^1_Y(\log \tilde{D})(U_2) = \left\{ p\left(\frac{z}{w}, w\right) \frac{d\left(\frac{z}{w}\right)}{\frac{z}{w}} + q\left(\frac{z}{w}, w\right) dw \,\middle|\, p, q \in \mathcal{O}_Z(V) \right\}$$

$$= \left\{ \frac{1}{z} p\left(\frac{z}{w}, w\right) dz + \left[q\left(\frac{z}{w}, w\right) - \frac{1}{w} p\left(\frac{z}{w}, w\right) \right] dw \,\middle|\, p, q \in \mathcal{O}_Z(V) \right\}.$$

These sections glue to a section of $\sigma_*(\Omega^1_Y(\log \tilde{D}))$ over V if coefficients of dz and dw are equal:

$$\frac{1}{z} g\left(z, \frac{w}{z}\right) = q\left(\frac{z}{w}, w\right) - \frac{1}{w} p\left(\frac{z}{w}, w\right) \tag{4.3}$$

$$\frac{1}{z}p\left(\frac{z}{w},w\right)=f\left(z,\frac{w}{z}\right)-\frac{w}{z^2}g\left(z,\frac{w}{z}\right) \tag{4.4}$$

Replacing $\frac{1}{z}g(z,\frac{w}{z})$ in Equation (4.4) with its equivalent expression coming from Equation (4.3) yields the equality

$$f\left(z,\frac{w}{z}\right)=\frac{w}{z}q\left(\frac{z}{w},w\right).$$

From this last expression, we see that

$$f\left(z,\frac{w}{z}\right)=\frac{1}{z}f'(z,w)$$

and

$$q\left(\frac{z}{w},w\right)=\frac{1}{w}f'(z,w)$$

where $f'(z,w)$ is a polynomial with $f'(0,0)=0$. Plugging these into Equation (4.3) and multiplying through by zw gives

$$wg\left(z,\frac{w}{z}\right)=z\left(f'\left(z,w\right)-p\left(\frac{z}{w},w\right)\right).$$

Since the right hand side is a polynomial in z, we can write $g\left(z,\frac{w}{z}\right)=zg'(z,w)$ for some polynomial g' with $g'(0,0)=0$, and rewrite the above equality as

$$wg'(z,w)=f'(z,w)-p\left(\frac{z}{w},w\right).$$

Therefore, $p\left(\frac{z}{w},w\right)=wg'(z,w)-f'(z,w)$. We now have expressions for f, g, p, and q as polynomials in z and w, which we can use in Equation (4.2). This gives us

$$\begin{aligned}\sigma_*(\Omega^1_Y(\log\tilde{D}))(V) &= \left\{\left[\frac{1}{z}f'(z,w)-\frac{w}{z}g'(z,w)\right]dz+g'(z,w)dw\,\middle|\,f',g'\in\mathcal{O}_Z(V)\right\}\\ &= \left\{f'(z,w)\frac{dz}{z}+g'(z,w)dw\,\middle|\,f',g'\in\mathcal{O}_Z(V)\right\}\end{aligned}$$

where the only restrictions on $f'(z,w)$ and $g'(z,w)$ are that neither has a constant term; that is, they both lie in the maximal ideal $\mathfrak{M}_p=(z,w)\subset\mathcal{O}_Z(V)\simeq\mathbb{C}[z,w]$. Thus,

$$\sigma_*(\Omega^1_Y(\log\tilde{D}))=\Omega^1_Z(\log D)\otimes\mathfrak{M}_p.$$

$\square$

Lemma 4.5. Let $f : X \to Y$ be a double cover of a smooth surface Y, and let B denote its smooth branch divisor. Let $C = f^{-1}(D)$ be the preimage of a smooth curve D on Y, and suppose that D intersects B transversally. Then

$$f_*(\Omega^1_X(\log C)) = \Omega^1_Y(\log D) \oplus \Omega^1_Y((\log D + B)(-L))$$

and

$$f_*(T_X(-\log C)) = T_Y(-\log(D+B)) \oplus T_Y(-\log D)(-L)$$

where $B \sim 2L$. Moreover, these decompositions break the sheaves into their invariant and anti-invariant subspace under the action of $\mathbb{Z}/2\mathbb{Z}$ by deck transformations.

Remark 4.6. Lemma 4.5 is an extension of the double cover version of [Par91, Lemma 4.2] to the log tangent sheaf.

Proof. In order to compute $f_*\Omega^1_X(\log C)$, note that it admits an action of $\mathbb{Z}/2\mathbb{Z}$ via deck transformations, so we can decompose it into its invariant and anti-invariant eigenspaces.

Let V be an open neighborhood of $p \in D \cap B$ and choose coordinates (z, w) on V so that p is at the origin and the local equation of D is z and the local equation of B is w. Then we have an open neighborhood U of $f^{-1}(p)$ with local coordinates (x, y) so that $f(x, y) = (x, y^2)$. Note that the ramification locus R of f has local equation y and the curve C on X has local equation x. See Figure 13.

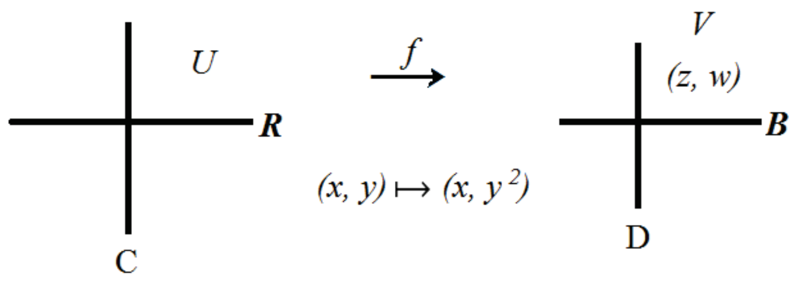


Figure 13. The map f.

On U we have

$$\Omega^1_X(\log C)(U) = \left\langle \frac{dx}{x}, dy \right\rangle_{\mathcal{O}_X(U)}.$$

Noting that $\mathcal{O}_Y(V) \simeq \mathbb{C}[x, y^2]$, we have

$$f_*(\Omega^1_X(\log C))(V) = \left\langle \frac{dx}{x}, y\frac{dx}{x}, dy, ydy \right\rangle_{\mathcal{O}_Y(V)}$$

The action of $\mathbb{Z}/2\mathbb{Z}$ sends (x, y) to $(x, -y)$. Therefore the invariant subspace of $f_*(\Omega^1_X(\log C))(V)$ is

$$\begin{aligned} f_*(\Omega^1_X(\log C))_+(V) &= \left\langle \frac{dx}{x}, ydy \right\rangle_{\mathcal{O}_Y(V)} \\ &= \left\langle \frac{dz}{z}, dw \right\rangle_{\mathcal{O}_Y(V)} \\ &= \Omega^1_Y(\log D)(V). \end{aligned}$$

The anti-invariant subspace of $f_*(\Omega^1_X(\log C))(V)$ is

$$\begin{aligned} f_*(\Omega^1_X(\log C))_-(V) &= \left\langle y\frac{dx}{x}, dy \right\rangle_{\mathcal{O}_Y(V)} \\ &= y\left\langle \frac{dx}{x}, \frac{dy}{y} \right\rangle_{\mathcal{O}_Y(V)} \\ &= y\left\langle \frac{dz}{z}, \frac{dw}{w} \right\rangle_{\mathcal{O}_Y(V)} \\ &= \Omega^1_Y((\log D + B)(-L))(V). \end{aligned}$$

One checks easily that these modules extend to the expected sheaves over all of Y. The proof for the log tangent bundle is similar. □

4.3 Smooth boundary components of $\overline{\mathcal{M}}_{5,5}$

We show that loci corresponding to surfaces of type 1 and 2a give generically smooth loci in the moduli space $\overline{\mathcal{M}}_{5,5}$. In both cases, we obtain this result by proving the vanishing of the cohomology group in which obstructions to $\mathbb{Q}$-Gorenstein deformations lie. Because the type 1 and 2a loci are 39-dimensional (see Theorem 3.6), we conclude that the closure of the 1 and 2a loci are generically smooth Cartier divisors in $\overline{\mathcal{M}}_{5,5}$.

4.3.1 The type 1 component

For this subsection, let W be a stable numerical quintic surface of type 1 or 1" and denote by S its minimal resolution. Let $f : S \to Z$ be the double cover, where $Z = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$, and f is branched over a smooth curve $B \sim 6\Delta$, tangent to $D \sim \Delta$ at six points. Then $f^*(D) = C_1 + C_2$ and the curves C_1 and C_2 are (-4) curves on S. Let $R = f^*B$ denote the ramification locus of f, and let $L \subset Z$ be a curve such that $B \sim 2L$.

In order to show that deformations of W are unobstructed, it suffices to show that $H^2(W, T_W) = 0$. Equivalently, as described above, we show that $H^2(S, T_S(-\log(C_1))) = 0$.

Theorem 4.7. Let S be the minimal resolution of a stable numerical quintic surfaces of type 1 or 1", and let C_1 and C_2 be the (-4)-curves on S. Then $H^2(S, T_S(-\log(C_1))) = H^2(S, T_S(-\log C_2)) = 0$.

Proof. Let $K = K_S$. We have

$$\Omega^1_S(\log(C_1))(K) \subset \Omega^1_S(\log(C_1 + C_2))(K).$$

Note that $K = f^*(K_Z + L)$. Since $K_Z \sim -2\Delta$ and $L \sim 3\Delta$, we have

$$K \sim f^*(\Delta) \sim f^*(D) = C_1 + C_2.$$

Since $C_1 + C_2 = f^*(D) \sim K$, we have

$$\Omega^1_S(\log(C_1 + C_2))(K) \subset \Omega^1_S(C_1 + C_2 + K) = \Omega^1_S(2K).$$

Ideally, we would like $H^0(S, \Omega^1_S(2K))$ to be zero, because Serre duality together with the above inclusion would imply that $H^2(S, T_S(-\log C_1)(K)) = 0$. We will use a different approach.

The double cover $f : S \to Z$ gives rise to an action of $\mathbb{Z}/2\mathbb{Z}$ on

$$H^0(S, \Omega^1_S(\log(C_1 + C_2))(K))$$

via deck transformations. This action interchanges C_1 and C_2. We claim that the groups $H^0(S, \Omega^1_S(\log C_1)(K))$ and $H^0(S, \Omega^1_S(\log C_2)(K))$ both lie in the anti-invariant subspace

$$H^0(S, \Omega^1_S \log(C_1 + C_2)(K))_-.$$

To see this, suppose that $\alpha \in \Omega^1_S(\log C_1)(K)$ is an invariant one-form. If α does not have a pole along C_1, then α is a global section of $\Omega^1_S(\log C_1)(K)$. But

$$H^0(S, \Omega^1_S(\log C_1)(K)) \simeq H^2(S, T_S)^\vee,$$

and by Horikawa [Hor76a], $H^2(S, T_S) = 0$. Thus, α has a pole along C_1. Then since the action of $\mathbb{Z}/2\mathbb{Z}$ interchanges C_1 and C_2, α must also have a pole along C_2. Therefore no such invariant one-form exists, so both cohomology groups $H^0(S, \Omega^1_S(\log C_1)(K))$ and $H^0(S, \Omega^1_S(\log C_2)(K))$ must both lie in the anti-invariant subspace

$$H^0(S, \Omega^1_S \log(C_1 + C_2)(K))_-.$$

We show that this subspace is zero.

By the projection formula, noting that $K \sim f^*(\Delta)$, we have

$$f_*(\Omega^1_S \log(C_1 + C_2)(K)) = f_*(\Omega^1_S \log(C_1 + C_2)(f^*\Delta)) = (f_*\Omega^1_S \log(C_1 + C_2))(\Delta).$$

We claim that

$$f_*\Omega^1_S(\log(C_1 + C_2))_- \subset \Omega_Z(\log B)(-2\Delta).$$

To compute $f_*(\Omega^1_S \log(C_1 + C_2))_-$, we need only consider a point in $C_1 \cap C_2 \cap R$. Indeed, suppose that U is a neighborhood of $p \in X$ such that $U \cap C_1 \cap C_2 \cap R = \emptyset$, and let V denote the image of U under f. By Lemma 4.5, we have

$$\begin{aligned} f_*(\Omega^1_S \log(C_1 + C_2))_-(V) &= \Omega^1_Z(\log(B + D))(-3\Delta)(V) \\ &= \Omega^1_Z(\log B)(-3\Delta)(V) \\ &\subset \Omega^1_Z(\log B)(-2\Delta)(V). \end{aligned}$$

where the second equality follows from the fact that $D \cap V = \emptyset$.

Now let U be an open subset of S containing $p \in C_1 \cap C_2 \cap R$, and let V an open neighborhood of $f(p)$. Choose coordinates (x, y) on U so that p is at the origin and the local equation of R is y. We can then choose coordinates (w, z) on V such that the local equation of B is z and the local equation of D is $z - w^2$. Then the local equations of C_1 and C_2 are $y - x$ and $y + x$. With these coordinates, the cover f is given by the function $(x, y) \mapsto (x, y^2)$. See Figure 14.

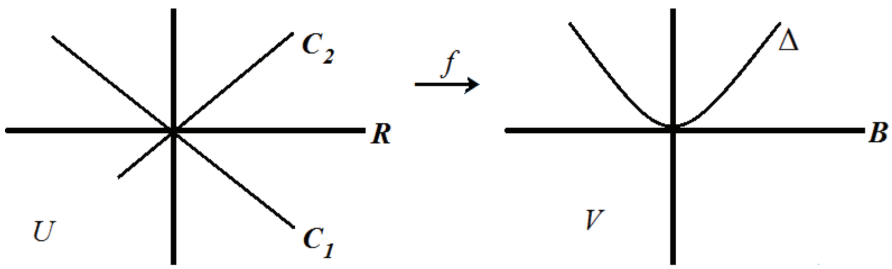


Figure 14. The map f.

The $\mathcal{O}_X(U)$-module $\Omega^1_S \log(C_1 + C_2)(U)$ is generated by $\left\{\frac{d(y-x)}{y-x}, \frac{d(y+x)}{y+x}\right\}$. As a module over $\mathcal{O}_Y(V)$, we have that $f_*\Omega^1_S \log(C_1 + C_2)(V)$ is generated by

$$\left\{\frac{d(y-x)}{y-x}, d(y-x), \frac{d(y+x)}{y+x}, d(y+x)\right\}$$

Since the action of $\mathbb{Z}/2\mathbb{Z}$ sends y to $-y$, we see quickly that the anti-invariant submodule is generated as an $\mathcal{O}_Y(V)$–module by

$$\begin{aligned}
\left\{\frac{d(y-x)}{y-x} + \frac{d(y+x)}{y+x}, dy\right\} &\subset \left\{\frac{1}{y^2 - x^2}(-2ydx + 2xdy), \frac{1}{y^2 - x^2}dy\right\} \\
&= \frac{1}{y^2 - x^2}\left\{(-2ydx + 2xdy), dy\right\} \\
&= \frac{y}{y^2 - x^2}\left\{-2dx, \frac{dy}{y}\right\} \\
&= \frac{y}{z - w^2}\left\{-2dw, \frac{dz}{z}\right\}
\end{aligned}$$

This last module we recognize as $\Omega^1_Z(\log B)(-3\Delta + D)(V) = \Omega^1_Z(\log B)(-2\Delta)(V)$. Thus,

$$f_*\Omega^1_S(\log(C_1 + C_2))_- \subset \Omega^1_Z(\log(B))(-2\Delta).$$

By the projection formula, using that $K \sim f^*\Delta$, we have

$$f_*\Omega^1_S(\log(C_1 + C_2))(K)_- \subset \Omega^1_Z(\log B)(-\Delta).$$

To show that $H^0(Z, \Omega^1_Z(\log B)(-\Delta)) = 0$, consider the exact sequence

$$0 \to \Omega^1_Z \to \Omega^1_Z(\log B) \to \mathcal{O}_B \to 0$$

where $\Omega^1_Z(\log B) \to \mathcal{O}_B$ is the residue map. Twisting by $-\Delta$ gives the exact sequence

$$0 \to \Omega^1_Z(-\Delta) \to \Omega^1_Z(\log B)(-\Delta) \to \mathcal{O}_B(-\Delta) \to 0.$$

Looking at the corresponding long exact sequence in cohomology, it remains to show that $H^0(Z, \Omega^1_Z(-\Delta)) = 0$ and $H^0(B, \mathcal{O}_B(-\Delta)) = 0$. Both of these are obvious, the first because $H^0(Z, \Omega^1_Z(-\Delta)) \subset H^0(Z, \Omega^1_Z) = 0$ and the second because $-\Delta \cdot B = -12 < 0$. $\square$

Remark 4.8. The proof of Theorem 4.7 extends easily to surfaces of type 1', which contain two (-4) curves C_1 and C_2 intersecting transversally at five points instead of six. To see this, suppose that F is the (-2) curve on S intersecting each of C_1 and C_2 transversally. The map f is then a double cover of $\tilde{Z}$, the blowup $\sigma : \tilde{Z} \to Z = \mathbb{P}^1 \times \mathbb{P}^1$ of a point $q \in D$. Let E denote the exceptional divisor of σ, and $\tilde{B}$, $\tilde{D}$, and $\tilde{L}$ the proper transforms of the curves $B \sim 6\Delta$, $D \sim \Delta$, and $L \sim 3\Delta$ in $Z = \mathbb{P}^1 \times \mathbb{P}^1$, respectively. Then f is branched over $\tilde{B}$, $f^*(\tilde{D}) = C_1 + C_2$, and $F = f^*E$.

We claim that

$$f_*(\Omega^1_S(\log(C_1 + C_2)))_- \subset \Omega_{\tilde{Z}}(\log \tilde{B})(-2\tilde{\Delta} - E),$$

where $\tilde{\Delta}$ is the proper transform of a generic curve Δ on $Z = \mathbb{P}^1 \times \mathbb{P}^1$. The same argument as above implies that this holds in a neighborhood of any point $p \notin F$, so it suffices to show the containment for a neighborhood U of $p \in F$. Since $p \notin C_1 \cap C_2$, Theorem 4.5 implies that, if $U = f^{-1}(V)$,

$$\begin{aligned}
f_*(\Omega^1_S(\log(C_1 + C_2)))_-(U) &= \Omega_{\tilde{Z}}(\log(\tilde{B} + \tilde{D}))(-\tilde{L})(V) \\
&= \Omega_{\tilde{Z}}(\log(\tilde{B} + \tilde{D}))(-3\tilde{D} - 3E)(V) \\
&= \Omega_{\tilde{Z}}(\log(\tilde{B}))(\tilde{D} - 3\tilde{D} - 3E)(V) \\
&= \Omega_{\tilde{Z}}(\log(\tilde{B}))(-2\tilde{D} - 3E)(V) \\
&= \Omega_{\tilde{Z}}(\log(\tilde{B}))(-2\tilde{\Delta} - E)(V)
\end{aligned}$$

where here we have used that $\tilde{D} \sim \tilde{\Delta} - E$ and $\tilde{L} \sim 3\tilde{D} + 3E$.

By the projection formula, we have

$$f_*(\Omega^1_S(\log(C_1 + C_2)))(K)_- = \Omega_{\tilde{Z}}(\log(\tilde{B}))(-\tilde{\Delta} - E).$$

To show vanishing of $H^0(\tilde{Z}, \Omega_{\tilde{Z}}(\log(\tilde{B}))(-\tilde{\Delta} - E))$, we use the exact sequence

$$0 \to \Omega^1_{\tilde{Z}}(-\tilde{\Delta} - E) \to \Omega^1_{\tilde{Z}}(\log \tilde{B})(-\tilde{\Delta} - E) \to \mathcal{O}_{\tilde{B}}(-\tilde{\Delta} - E) \to 0.$$

The result then follows from the long exact sequence in cohomology, because

$$\tilde{B} \cdot (-\tilde{\Delta} - E) = -12 - 2 = 14 < 0$$

and

$$H^0(\tilde{Z}, \Omega^1_{\tilde{Z}}(-\tilde{\Delta} - E)) \subset H^0(\tilde{Z}, \Omega^1_{\tilde{Z}}) = 0.$$

In Section 3.3, we showed that the locus of stable quintic surfaces of type 1 is 39-dimensional, so Theorems 4.1 and 4.7 imply the following:

Corollary 4.9. The closure of the locus of surfaces of type 1 is a generically smooth Cartier divisor in $\overline{\mathcal{M}}_{5,5}$, lying in the closure of the type I component of $\mathcal{M}_{5,5}$.

4.3.2 The 2a component

Let W be a stable numerical quintic surface of type 2a, 2a', or 2a" and let S denote its minimal resolution. Then there is a map $\tilde{f} : S \to \tilde{Z}$, which is the double cover of the blowup of $Z = \mathbb{P}^1 \times \mathbb{P}^1$ in two points p and q lying on a fiber D. The branch locus $\tilde{B}$ of $\tilde{f}$ is the proper transform of an irreducible curve $B \sim 6\Delta$ which has either a node or an A_2 singularity at each of p and q and is smooth elsewhere. Denote by Γ_1 and Γ_2 generic rulings of $\tilde{Z}$ so that $\Gamma_2 \sim \tilde{D} + E_1 + E_2$, where $\tilde{D}$ is the proper transform of $D \subset Z$.

Theorem 4.10. Let W be a stable numerical quintic surface of type 2a, 2a', or 2a", let S be its minimal resolution and C the (-4)-curve on S. Then $H^2(S, T_S(-\log C))) = 0$.

We begin with a lemma.

Lemma 4.11. $H^0(\tilde{Z}, \Omega^1_{\tilde{Z}}(\log \tilde{D} + \tilde{B})(K_{\tilde{Z}})) = 0.$

Proof. We have the following exact sequence of sheaves on $\tilde{Z}$:

$$0 \to \Omega^1_{\tilde{Z}} \to \Omega^1_{\tilde{Z}}(\log(\tilde{D} + \tilde{B})) \to \mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}} \to 0$$

where $\Omega^1_{\tilde{Z}}(\log \tilde{D} + \tilde{B}) \to \mathcal{O}_{\tilde{D}+\tilde{B}}$ is the residue map. Twisting by $K_{\tilde{Z}}$ gives the exact sequence

$$0 \to \Omega^1_{\tilde{Z}}(K_{\tilde{Z}}) \to \Omega^1_{\tilde{Z}}(\log \tilde{D} + \tilde{B})(K_{\tilde{Z}}) \to (\mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}})(K_{\tilde{Z}}) \to 0. \tag{4.5}$$

Note that

$$K_{\tilde{Z}} = \sigma^*(K_{\mathbb{P}^1\times\mathbb{P}^1}) + E_1 + E_1 = -2\Gamma_1 - 2\Gamma_2 + E_1 + E_2 \sim -2\Gamma_1 - 2\tilde{D} - E_1 - E_2, \tag{4.6}$$

and so $-K_{\tilde{Z}}$ is effective. Thus $H^0(\tilde{Z}, \Omega^1_{\tilde{Z}}(K_{\tilde{Z}})) \subset H^0(\tilde{Z}, \Omega^1_{\tilde{Z}})$. Since the irregularity of $\tilde{Z}$ is zero, we have $H^0(\tilde{Z}, \Omega^1_{\tilde{Z}})(K_{\tilde{Z}}) = 0$. Moreover, noting that $\sigma^*(B) = \tilde{B} + 2E_1 + 2E_2$ and $\sigma^*(K_Z) = K_{\tilde{Z}} - E_1 - E_2$, we have

$$K_{\tilde{Z}} \cdot \tilde{B} = -24 < 0$$

and

$$K_{\tilde{Z}} \cdot \tilde{D} = 0.$$

Therefore $H^0(\tilde{Z}, (\mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}})(K_{\tilde{Z}})) = \mathbb{C}$, so the cohomology group $H^0(\tilde{Z}, \Omega^1_{\tilde{Z}}(\log(\tilde{D} + \tilde{B}))(K_{\tilde{Z}}))$ is 0 if and only if the connecting homomorphism

$$\delta : H^0(\tilde{Z}, (\mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}})(K_{\tilde{Z}})) \to H^1(\tilde{Z}, \Omega^1_{\tilde{Z}}(K_{\tilde{Z}}))$$

is injective.

Since $-K_{\tilde{Z}}$ is effective, we have a section $s \in H^0(\tilde{Z}, \Omega^1_{\tilde{Z}}(-K_{\tilde{Z}}))$, so we have a map from the short exact sequence (4.5) to the short exact sequence

$$0 \to \Omega^1_{\tilde{Z}} \to \Omega^1_{\tilde{Z}}(\log(\tilde{D} + \tilde{B})) \to \mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}} \to 0.$$

where the map is given by tensoring with s. The connecting homomorphism

$$\delta_2 : H^0(\tilde{Z}, \mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}}) \to H^1(\tilde{Z}, \Omega^1_{\tilde{Z}})$$

of the corresponding short exact sequence is the first Chern class map. That is, if $1_{\tilde{D}}$ and $1_{\tilde{B}}$ are generators of $H^0(\tilde{Z}, \mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}})$, then $\delta_2(1_{\tilde{D}}) = c_1(\tilde{D}) = D$ and $\delta_2(1_{\tilde{B}}) = c_1(B)$. Thus, the map δ_2 is injective if and only if the curves $\tilde{D}$ and $\tilde{B}$ are linearly independent in the Picard group of $\tilde{Z}$. Recalling that $\mathrm{Pic}(\tilde{Z})$ is generated by Γ_1, Γ_2, E_1 and E_2, and that $\tilde{B} \sim 6\Gamma_1 + 6\Gamma_2 - 2E_1 - 2E_2$ and $\tilde{D} \sim \Gamma_2 - E_1 - E_2$, we see that the two divisors are indeed linearly independent.

Thus, we have a diagram

$$\begin{tikzcd}
H^0(\tilde{Z}, (\mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}})(K_{\tilde{Z}})) \arrow[r, "\delta"] \arrow[d, "\otimes s"'] & H^1(\tilde{Z}, \Omega^1_{\tilde{Z}}(K_{\tilde{Z}})) \arrow[d, "\otimes s"] \\
H^0(\tilde{Z}, \mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}}) \arrow[r, "\delta_2"] & H^1(\tilde{Z}, \Omega^1_{\tilde{Z}})
\end{tikzcd}$$

where the bottom arrow is injective. We see that δ is injective as long as the map on the left is injective. But this map simply takes a section of $(\mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}})(K_{\tilde{Z}})$ and multiplies it by s. Since $s \neq 0$, the map is injective. □

Proof of Theorem 4.10. We show that $H^2(S, T_S(-\log C)) = 0$, where S is the minimal resolution of W and C is the (-4)-curve on S. By Serre duality, it is enough to show that $H^0(S, \Omega^1_S(\log C)(K_S)) = 0$. Recall that $C = f^*\tilde{D}$ and $K_S = f^*(K_Y + \tilde{L})$. By the projection formula

$$f_*(\Omega^1_S(\log C)(K_S)) = (f_*\Omega^1_S(\log C)) \otimes (K_Y + \tilde{L}).$$

Together with Lemma 4.5, this gives us

$$f_*(\Omega^1_S(\log C)(K_S)) = \Omega^1_Y(\log \tilde{D})(K_Y + \tilde{L}) \oplus \Omega^1_Y(\log \tilde{D} + \tilde{B})(K_Y).$$

By Lemma 4.11, we have $H^0(Y, \Omega^1_Y(\log \tilde{D} + \tilde{B})(K_Y)) = 0$. It remains to show that $H^0(Y, \Omega^1_Y(\log \tilde{D})(K_Y + \tilde{L})) = 0$, which we do via the projection formula. By Lemma 4.4, we have $\sigma_*\Omega^1_Y(\log \tilde{D}) = \Omega^1_Z(\log D) \otimes \mathfrak{M}_{p,q}$, where $\mathfrak{M}_{p,q}$ is the ideal sheaf of p and q which are the centers of σ. Noting that $(K_Y + \tilde{L}) = f^*(\Delta)$, the projection formula gives

$$\begin{aligned}
\sigma_*(\Omega^1_Y(\log \tilde{D})(K_Y + \tilde{L})) &= (\Omega^1_{\mathbb{P}^1\times\mathbb{P}^1}(\log D) \otimes \mathfrak{M}_{p,q}) \otimes \mathcal{O}(\Delta) \\
&= (\Omega^1_{\mathbb{P}^1\times\mathbb{P}^1}(\log D) \otimes \mathfrak{M}_{p,q}) \otimes \mathcal{O}(\Delta) \\
&= [(p_1^*\Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathfrak{M}_{p,q}) \oplus (p_2^*\Omega^1_{\mathbb{P}^1} \otimes \mathfrak{M}_{p,q})] \otimes \mathcal{O}(\Delta) \\
&= [\mathcal{O}(-1,0) \otimes \mathcal{O}(1,1) \otimes \mathfrak{M}_{p,q}] \oplus [\mathcal{O}(0,-2) \otimes \mathcal{O}(1,1) \otimes \mathfrak{M}_{p,q}] \\
&= (\mathcal{O}(0,1) \otimes \mathfrak{M}_{p,q}) \oplus (\mathcal{O}(1,-1) \otimes \mathfrak{M}_{p,q}).
\end{aligned}$$

We have $H^0(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}(1,-1)\otimes\mathfrak{M}_{p,q}) = 0$, because $H^0(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}(a,b)) = 0$ for $a < 0$ or $b < 0$. And $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(0,1) \otimes \mathfrak{M}_{p,q}) = 0$, since p and q lie on $D \in |1,0|$. □

By Theorem 3.6, the locus of 2a surfaces is 39-dimensional. Moreover, Theorem 4.2 shows that every 2a surfaces may be obtained as the stable limit of a family of numerical quintic surfaes of type IIa. Together with Theorem 4.10, this implies the following

Corollary 4.12. The closure of the locus of surfaces of type 2a is a generically smooth Cartier divisor in $\overline{\mathcal{M}}_{5,5}$, lying in the closure of the type IIa component of $\mathcal{M}_{5,5}$.

CHAPTER 5

DEFORMATIONS OF 2b SURFACES

We study the versal $\mathbb{Q}$-Gorenstein deformation space $\mathrm{Def}^{QG}(W)$ [Hac04] where W is general 2b surface. All deformation functors considered are functors of Artinian rings. However, because W is a stable surface, we often abuse notation and view $\mathrm{Def}^{QG}(W)$ as an analytic germ of a point $[W]$ in the KSBA moduli space. The same notational ambiguity applies to other deformation functors we consider which admit a moduli space. This enables us to study the moduli space $\overline{\mathcal{M}}$ using analytic methods of Horikawa [Hor75, Hor76a]. The main theorem is

Theorem 5.1. The locus of stable numerical quintic surfaces whose unique non Du Val singularity is a $\frac{1}{4}(1,1)$ singularity forms a divisor in $\overline{\mathcal{M}}_{5,5}$ which consists of two 39-dimensional components $\bar{1}$ and $\overline{2\mathrm{a}}$ meeting, transversally at a general point, in a 38-dimensional component $\overline{2\mathrm{b}}$. This divisor is Cartier at general points of the $\bar{1}$, $\overline{2\mathrm{a}}$, and $\overline{2\mathrm{b}}$ components. These components are the closures of the loci of 1, 2a, and 2b surfaces described at the beginning of Section 3.3. Moreover, the type $\bar{1}$, $\overline{2\mathrm{a}}$, and $\overline{2\mathrm{b}}$ components belong to the closure of the components in $\mathcal{M}_{5,5}$ of types I, IIa, and IIb, respectively.

The proof will consist of several pieces. Theorems 4.7 and 4.10 showed that obstructions to deformations of surfaces of types 1 and 2a vanish, and so the closures of their corresponding 39-dimensional loci in $\overline{\mathcal{M}}_{5,5}$ are generically smooth Cartier divisors. In Section 5.1, Theorem 5.2, we show that deformations of 2b surfaces are obstructed and that the obstruction space is one-dimensional. This implies that the space $\mathrm{Def}^{QG}(W)$ of

$\mathbb{Q}$-Gorenstein deformations of a generic 2b surface W is a hypersurface in some ambient space.

By Theorem 4.2, there exists a $\mathbb{Q}$-Gorenstein smoothing of a 2b surface to a numerical quintic surface of type IIb which induces a versal deformation of the singularity. Therefore, the map $\mathrm{Def}^{QG}(W) \to \mathrm{Def}_p^{QG,\mathrm{loc}}$ to local $\mathbb{Q}$-Gorenstein deformations of the $\frac{1}{4}(1,1)$ singularity ($p \in W$) is a submersion. Since this latter space is one-dimensional, this implies that the space $\mathrm{Def}^{QG}(W)$ is analytically isomorphic to $\mathrm{Def}^{QG,\mathrm{e.s.}}(W) \times \mathbb{A}^1$, where $\mathrm{Def}^{QG,\mathrm{e.s.}}(W)$ is the space of equisingular $\mathbb{Q}$-Gorenstein deformations of W. Therefore, the description of $\mathrm{Def}^{QG}(W)$ is complete as soon as we can describe the space $\mathrm{Def}^{QG,\mathrm{e.s.}}(W)$. Moreover, the space $\mathrm{Def}^{QG,\mathrm{e.s.}}(W)$ is isomorphic to the deformation space of pairs $\mathrm{Def}(S,C)$, where S is the minimal resolution of W, containing (-4)-curve C.

In Section 5.2, we describe a subfunctor of the deformation functor of pairs $\mathcal{D}ef\,(S,C)$, and show that it has no obstructions. This will imply that the space $\mathrm{Def}(S,C)$ contains a smooth component corresponding to the 2a locus. Thus, to prove that $\mathrm{Def}^{QG,\mathrm{e.s.}}$ is a union of two 39-dimensional components meeting transversally in a 38-dimensional component, it suffices to show that the degree two part of the Kuranishi map, given by the Schouten bracket, is nonzero and not a square. Horikawa makes a similar argument in [Hor75] and [Hor76a]. In Sections 5.3 and 5.4, we extend his work to the log setting.

It will be useful to understand the Kuranishi deformation space in more generality. Suppose that S is a smooth surface, and let $\mathrm{Def}(S)$ be the space of deformations of S. The tangent space to $\mathrm{Def}(S)$, that is the space of first order infinitesimal deformations of S, is isomorphic via the Kodaira–Spencer map to the cohomology group $H^1(S,T_S)$. Let $\rho_1,\dots,\rho_n$ be a basis of $H^1(S,T_S)$, and let $t_1,\dots t_n$ be a dual basis. Then $\mathrm{Def}(S)$ is locally analytically isomorphic to a subspace of $\mathbb{C}^{40}$ with coordinates $t_1,\dots,t_{40}$, and is given by the kernel of the Kuranishi map $k : H^1(S,T_S) \to H^2(S,T_S)$, which is a certain infinite series in $t_1,\dots t_n$. Catanese's article [Cat] gives an excellent exposition of the construction of the Kuranishi map. For us, the important part is that the degree two part of the Kuranishi map is given by the Schouten bracket, which we now describe.

The Schouten bracket is the bilinear map

$$[,] : H^1(S, T_S) \otimes H^1(S, T_S) \to H^2(S, T_S)$$

defined as the composition of the cup product $\cup : H^1(S, T_S) \otimes H^1(S, T_S) \to H^2(S, T_S \otimes T_S)$ followed by the Lie bracket $H^2(S, T_S \otimes T_S) \to H^2(S, T_S)$. If S_ρ is the infinitesimal first order deformation corresponding, via the Kodaira–Spencer map, to $\rho \in H^1(S, T_S)$, then $[\rho, \rho]$ is the cohomology class corresponding to the obstruction to extending the deformation S_ρ to the second order. More explicitly, the Schouten bracket is defined in coordinates as follows: let $\{U_i\}$ is a sufficiently fine open covering of S and let $U_{ij} = U_i \cap U_j$. Let $z_i = (z_i^1, z_i^2)$ be holomorphic coordinates on U_i such that $z_i = b_{ij}(z_j)$ on U_{ij}, where b_{ij} are holomorphic functions. If the element $\rho \in H^1(S, T_S)$ is represented by the one-cocycle $\{\rho_{ij}\}$, then the first-order deformation S_ρ of S has holomorphic coordinates on U_i given by

$$\phi_i = b_{ij}(z_j) + \rho_{ij} t.$$

On $U_{ij} \cap U_{jk}$, the class $[\rho, \rho] \in H^2(S, T_S)$ is represented by the 2-cocycle $\{\xi_{ijk}\}$ that is given by the Lie bracket $[\rho_{ij}, \rho_{jk}]$. If $[\rho, \rho] = 0$, then the first-order deformation extends to the second order in t as

$$\phi_i = b_{ij}(z_j) + \rho_{ij} t + \tilde{\rho}_{ij} t^2.$$

where $\{\tilde{\rho}_{ij}\}$ is a one-cochain with coefficients in T_S whose Čech differential gives the two-cocycle $\{\xi_{ijk}\}$.

We use the following notation throughout this chapter. Let S be the minimal resolution of a surface of type 2b. We recall the construction of S. Let $\sigma : \tilde{\mathbb{F}}_2 \to \mathbb{F}_2$ be the blowup of $\mathbb{F}_2$ in two distinct points p and q lying on a fiber D. Denote by $\tilde{D}$ and Γ the proper transforms of D and a generic fiber, respectively, and let E_1 and E_2 be the exceptional divisors of σ. By abuse of notation, we denote by Δ_0 the proper transform of the negative section Δ_0 on $\mathbb{F}_2$. Let B be a reduced, irreducible divisor in the linear system $|6\Delta_0 + 12\Gamma|$ on $\mathbb{F}_2$ which is smooth away from p and q and with simple nodes at p and q. Let $\tilde{B}$ be its proper transform and note that $\tilde{B} \sim 2\tilde{L}$ for some smooth divisor L on $\tilde{\mathbb{F}}_2$. Then S is given by the double cover of $f : S \to \tilde{\mathbb{F}}_2$ branched over $\tilde{B}$. The curve C given by $f^*(\tilde{D})$ is the (-4)-curve on S. Moreover S contains four (-2)-curves: F_1 and F_2 mapping to Δ_0, and $\bar{E}_1$ and $\bar{E}_2$ mapping to E_1 and E_2, respectively. We denote by $\pi : \mathbb{F}_2 \to \mathbb{P}^1$ and $g : S \to \mathbb{P}^1$ the projection maps to $\mathbb{P}^1$.

5.1 The obstruction

To begin with, we show that the obstruction space is one-dimensional.

Theorem 5.2. Let S be the minimal resolution of a 2b surface, and let C denote the (-4)-curve on S. Then $H^2(S, T_S(-\log C)) = \mathbb{C}$.

The proof of Theorem 5.2 requires two lemmas.

Lemma 5.3. Let $Z = \mathbb{F}_2$ and $\tilde{Z}$ the blowup of Z in p and q. Then $H^0(\tilde{Z}, \Omega^1_{\tilde{Z}}(\log(\tilde{D} + \tilde{B} + \Delta_0))(K_{\tilde{Z}})) = 0$.

Proof. The proof is very similar to that of Lemma 4.11.

We have the following exact sequence of sheaves on $\tilde{Z}$:

$$0 \to \Omega^1_{\tilde{Z}}(K_{\tilde{Z}}) \to \Omega^1_{\tilde{Z}}(\log(\tilde{D} + \tilde{B} + \Delta_0))(K_{\tilde{Z}}) \to (\mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}} \oplus \mathcal{O}_{\Delta_0})(K_{\tilde{Z}}) \to 0.$$

where $\Omega^1_{\tilde{Z}}(\log \tilde{D} + \tilde{B}) \to \mathcal{O}_{\tilde{D}+\tilde{B}}$ is the residue map. Twisting by $K_{\tilde{Z}}$ gives the exact sequence

$$0 \to \Omega^1_{\tilde{Z}}(K_{\tilde{Z}}) \to \Omega^1_{\tilde{Z}}(\log \tilde{D} + \tilde{B} + \Delta_0)(K_{\tilde{Z}}) \to (\mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}} \oplus \mathcal{O}_{\Delta_0})(K_{\tilde{Z}}) \to 0. \tag{5.1}$$

Note that

$$\begin{aligned} K_{\tilde{Z}} &= \sigma^*(K_{\mathbb{F}_2}) + E_1 + E_1 \\ &= -2\Delta_0 - 4\Gamma + E_1 + E_2 \\ &\sim -2\Delta_0 - 4\tilde{D} - 3E_1 - 3E_2, \end{aligned}$$

and so $-K_{\tilde{Z}}$ is effective. Thus $H^0(\tilde{Z}, \Omega^1_{\tilde{Z}}(K_{\tilde{Z}})) \subset H^0(\tilde{Z}, \Omega^1_{\tilde{Z}})$. Since the irregularity of $\tilde{Z}$ is zero, we have $H^0(\tilde{Z}, \Omega^1_{\tilde{Z}})(K_{\tilde{Z}}) = 0$. Moreover, noting that $\sigma^*(B) = \tilde{B} + 2E_1 + 2E_2$ and $\sigma^*(K_Z) = K_{\tilde{Z}} - E_1 - E_2$, we have

$$K_{\tilde{Z}} \cdot \tilde{B} = -24 < 0,$$

$$K_{\tilde{Z}} \cdot \tilde{D} = 0,$$

and

$$K_{\tilde{Z}} \cdot \Delta_0 = 0.$$

Therefore $H^0(\tilde{Z}, (\mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}} \oplus \mathcal{O}_{\Delta_0})(K_{\tilde{Z}})) = \mathbb{C}^2$, so the cohomology group $H^0(\tilde{Z}, \Omega^1_{\tilde{Z}}(\log(\tilde{D} + \tilde{B} + \Delta_0))(K_{\tilde{Z}}))$ is 0 if and only if the connecting homomorphism

$$\delta : H^0(\tilde{Z}, (\mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}} \oplus \mathcal{O}_{\Delta_0})(K_{\tilde{Z}})) \to H^1(\tilde{Z}, \Omega^1_{\tilde{Z}}(K_{\tilde{Z}}))$$

is injective.

Since $-K_{\tilde{Z}}$ is effective, we have a section $s \in H^0(\tilde{Z}, \Omega^1_{\tilde{Z}}(-K_{\tilde{Z}}))$, so we have a map from the short exact sequence (5.1) to the short exact sequence

$$0 \to \Omega^1_{\tilde{Z}} \to \Omega^1_{\tilde{Z}}(\log(\tilde{D} + \tilde{B} + \Delta_0)) \to \mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B} \oplus \mathcal{O}_{\Delta_0}} \to 0.$$

where the map is given by tensoring with s. The connecting homomorphism

$$\delta_2 : H^0(\tilde{Z}, \mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}} \oplus \mathcal{O}_{\Delta_0}) \to H^1(\tilde{Z}, \Omega^1_{\tilde{Z}})$$

of the corresponding short exact sequence is the first Chern class map. That is, if $1_{\tilde{D}}$, $1_{\tilde{B}}$, and 1_{Δ_0} are generators of $H^0(\tilde{Z}, \mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}} \oplus \mathcal{O}_{\Delta_0})$, then $\delta_2(1_{\tilde{D}}) = c_1(\tilde{D}) = \tilde{D}$,

$\delta_2(1_{\tilde{B}}) = c_1(\tilde{B}) = \tilde{B}$, and $\delta_2(1_{\Delta_0}) = c_1(\Delta_0) = \Delta_0$. Thus, the map δ_2 is injective if and only if the curves $\tilde{D}$, $\tilde{B}$, and Δ_0 are linearly independent in the Picard group of $\tilde{Z}$. Recalling that $\mathrm{Pic}(\tilde{Z})$ is generated by Δ_0, Γ, E_1 and E_2, that $\tilde{B} \sim 6\Delta_0 + 12\Gamma - 2E_1 - 2E_2$ and that $\tilde{D} \sim \Gamma - E_1 - E_2$, we see that the three divisors are indeed linearly independent.

Thus, we have a diagram

$$\begin{tikzcd}
H^0(\tilde{Z}, (\mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}} \oplus \mathcal{O}_{\Delta_0})(K_{\tilde{Z}})) \arrow[r, "\delta"] \arrow[d, "\otimes s"'] & H^1(\tilde{Z}, \Omega^1_{\tilde{Z}}(K_{\tilde{Z}})) \arrow[d, "\otimes s"] \\
H^0(\tilde{Z}, \mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}} \oplus \mathcal{O}_{\Delta_0}) \arrow[r, "\delta_2"] & H^1(\tilde{Z}, \Omega^1_{\tilde{Z}})
\end{tikzcd}$$

where the bottom arrow is injective. We see that δ is injective as long as the map on the left is injective. But this map simply takes a section of $(\mathcal{O}_{\tilde{D}} \oplus \mathcal{O}_{\tilde{B}})(K_{\tilde{Z}})$ and multiplies it by s. Since $s \neq 0$, the map is injective. $\square$

Lemma 5.4. $H^0(\tilde{\mathbb{F}}_2, \Omega^1_{\tilde{\mathbb{F}}_2}(\log \tilde{D})(K_{\tilde{\mathbb{F}}_2} + \tilde{L})) = \mathbb{C}$.

Proof. By the projection formula we have

$$\begin{aligned}
\sigma_*(\Omega^1_{\tilde{\mathbb{F}}_2}(\log \tilde{D})(K_{\tilde{\mathbb{F}}_2} + \tilde{L})) &= \sigma_*(\Omega^1_{\tilde{\mathbb{F}}_2}(\log \tilde{D}))(K_{\mathbb{F}_2} + L) \\
&= \sigma_*(\Omega^1_{\tilde{\mathbb{F}}_2}(\log \tilde{D})) \otimes \mathcal{O}(\Delta_0 + 2\Gamma).
\end{aligned}$$

Lemma 4.4 gives

$$\sigma_*(\Omega^1_{\tilde{\mathbb{F}}_2}(\log \tilde{D})) = \Omega^1_{\mathbb{F}_2}(\log D) \otimes \mathfrak{M}_{p,q}.$$

Thus,

$$\sigma_*(\Omega^1_{\tilde{\mathbb{F}}_2}(\log \tilde{D})(K_{\tilde{\mathbb{F}}_2} + \tilde{L})) = \Omega^1_{\mathbb{F}_2}(\log D) \otimes \mathfrak{M}_{p,q} \otimes \mathcal{O}(\Delta_0 + 2\Gamma).$$

Let $\pi : \mathbb{F}_2 \to \mathbb{P}^1$ be the projection map, and suppose that $\pi(D) = a$. We have the short exact sequence

$$0 \to \pi^*\Omega^1_{\mathbb{P}^1}(\log a) \to \Omega^1_{\mathbb{F}_2}(\log D) \to \mathcal{O}_{\mathbb{F}_2}(-2\Delta_0 - 2\Gamma) \to 0.$$

The sheaf $\mathcal{O}_{\mathbb{F}_2}(-2\Delta_0 - 2\Gamma)$ is free, so $\mathrm{Tor}_1(\mathfrak{M}_{p,q} \otimes \mathcal{O}(\Delta_0 + 2\Gamma), \mathcal{O}_{\mathbb{F}_2}(-2\Delta_0 - 2\Gamma)) = 0$. Thus, tensoring $\mathfrak{M}_{p,q} \otimes \mathcal{O}(\Delta_0 + 2\Gamma)$ with the above short exact sequence yields the new short exact sequence

$$
\begin{aligned}
0 \to \pi^*\Omega^1_{\mathbb{P}^1}(\log a) \otimes \mathfrak{M}_{p,q} \otimes \mathcal{O}(\Delta_0 + 2\Gamma) \quad &\to \quad \Omega^1_{\mathbb{F}_2}(\log D) \otimes \mathfrak{M}_{p,q} \otimes \mathcal{O}(\Delta_0 + 2\Gamma) \\
&\to \quad \mathcal{O}_{\mathbb{F}_2}(-\Delta_0) \otimes \mathfrak{M}_{p,q} \to 0.
\end{aligned}
$$

Since $\mathbb{F}_2$ is projective, the sheaf $\mathcal{O}_{\mathbb{F}_2}(-\Delta_0) \otimes \mathfrak{M}_{p,q}$ has no global holomorphic sections, and so

$$
H^0(\mathbb{F}_2, \pi^*\Omega^1_{\mathbb{P}^1}(\log a) \otimes \mathfrak{M}_{p,q} \otimes \mathcal{O}(\Delta_0 + 2\Gamma)) \cong H^0(\mathbb{F}_2, \Omega^1_{\mathbb{F}_2}(\log D) \otimes \mathfrak{M}_{p,q} \otimes \mathcal{O}(\Delta_0 + 2\Gamma)).
$$

The sheaf $\Omega^1_{\mathbb{P}^1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2)$, so by the projection formula we have

$$
H^0(\mathbb{F}_2, \pi^*\Omega^1_{\mathbb{P}^1}(\log a) \otimes \mathfrak{M}_{p,q} \otimes \mathcal{O}(\Delta_0 + 2\Gamma)) = H^0(\mathbb{F}_2, \mathcal{O}_{\mathbb{F}_2}(\Delta_0 + \Gamma) \otimes \mathfrak{M}_{p,q}).
$$

The divisor Δ_0 satisfies $\Delta_0 \cdot (\Delta_0 + \Gamma) = -1$, so that Δ_0 is a fixed part of the linear system $|\Delta_0 + \Gamma|$. Since D is the only fiber containing both p and q, this implies that

$$
H^0(\mathbb{F}_2, \mathcal{O}_{\mathbb{F}_2}(\Delta_0 + \Gamma) \otimes \mathfrak{M}_{p,q}) = \mathbb{C},
$$

completing the proof. $\square$

We can now prove the theorem.

Proof of Theorem 5.2. By Serre duality, it suffices to show that

$$
H^0(S, \Omega^1_S(\log C)(K_S)) = \mathbb{C}.
$$

By Lemma 4.5 and the projection formula, we have

$$
f_*(\Omega^1_S(\log C)(K_S)) = \Omega^1_{\tilde{\mathbb{F}}_2}(\log \tilde{D})(K_{\tilde{\mathbb{F}}_2} + \tilde{L}) \oplus \Omega^1_{\tilde{\mathbb{F}}_2}(\log(\tilde{D} + \tilde{B}))(K_{\tilde{\mathbb{F}}_2}).
$$

By Lemma 5.4, we have $H^0(\Omega^1_{\tilde{\mathbb{F}}_2}(\log \tilde{D})(K_{\tilde{\mathbb{F}}_2} + \tilde{L})) = \mathbb{C}$. Moreover,

$$
\Omega^1_{\tilde{\mathbb{F}}_2}(\log(\tilde{D} + \tilde{B}))(K_{\tilde{\mathbb{F}}_2}) \subset \Omega^1_{\tilde{\mathbb{F}}_2}(\log(\tilde{D} + \tilde{B} + \Delta_0))(K_{\tilde{\mathbb{F}}_2}).
$$

Thus, $H^0(\Omega^1_{\tilde{\mathbb{F}}_2}(\log(\tilde{D} + \tilde{B}))(K_{\tilde{\mathbb{F}}_2})) = 0$ by Lemma 5.3. $\square$

5.2 Deformations of pairs and the equisingular locus

Let $f : X \to Y$ be the double cover of a smooth surface Y branched over a smooth curve B. Define $\mathrm{Def}_{X\to Y}$ to be the space of deformations of X that are double covers of deformations of Y. The group $\mathbb{Z}/2\mathbb{Z}$ acts on X by deck transformations, and the sheaf f_*T_X decomposes into invariant and anti-invariant subspaces as

$$f_*T_X \simeq T_Y(-\log B) \oplus T_Y(-L),$$

where $2L \sim B$ [Par91].

Theorem 5.5. [CvS06] Via the decomposition of f_*T_X into its invariant and anti-invariant subspaces, the deformation space $\mathrm{Def}(X \to Y)$ of double covers of deformations of Y may be identified with the deformation space $\mathrm{Def}(Y, B)$ of deformations of pairs, where B is the branch divisor of f.

The proof of Theorem 5.5 involves identifying the space of infinitesimal deformations of double covers of deformations of Y with the anti-invariant subspace $H^1_+(X, T_X) \subset H^1(X, T_X)$. Then using the decomposition of $f_*(T_X)$ above, this space is isomorphic to $H^1(Y, T_Y(-\log B))$.

Using Lemma 4.5, the same analysis works in the presence of the curves $C \subset X$ and $D \subset Y$, as long as D intersects B transversally. More explicitly, define $\mathcal{D}ef_{(X,C)\to(Y,D)}$ to be the functor of Artinian local rings which associates to an Artinian local ring A the set of isomorphism classes of deformations over A of squares

$$\begin{tikzcd} X \arrow[r] & Y \\ C \arrow[u, hook] \arrow[r] & D \arrow[u, hook] \end{tikzcd}$$

where the top and bottom maps are double covers and the left and right maps are embeddings of the smooth curves C and D into X and Y, respectively. Then the functor $\mathcal{D}ef_{(X,C)\to(Y,D)}$ may be identified with the functor $\mathcal{D}ef_{(Y,B,D)}$ of deformations of triples. The space of first-order infinitesimal deformations of triples (Y, B, D) is therefore $H^1_+(X, T_X(-\log C))$. By Lemma 4.5, we have

$$H^1_+(X, T_X(-\log C)) \simeq H^1(Y, T_Y(-\log(B + D))).$$

There is a forgetful map $\alpha : \mathcal{D}ef_{(X,C)\to(Y,D)} \to \mathcal{D}ef_{(X,C)}$. This map is an analytic embedding, because the differential

$$d\alpha : H^1(Y, T_Y(-\log(B + D))) \to H^1_+(X, T_X(-\log C)) \subset H^1(X, T_X(-\log C))$$

is an isomorphism onto its image.

Suppose now that W is a stable numerical quintic surface of type 2b, S its minimal resolution, and C the (-4) curve on S. Then we have the commutative square

$$\begin{tikzcd} S \arrow[r, "\tilde{f}"] & \tilde{Z} \\ C \arrow[u, hook] \arrow[r] & \tilde{D} \arrow[u, hook] \end{tikzcd}$$

where $\tilde{f}$ is the double cover of $\tilde{Z}$, where $\tilde{Z}$ is the blowup of $Z = \mathbb{F}_2$ in two points lying on a fiber D. The branch curve of $\tilde{f}$ is a smooth curve $\tilde{B}$, which intersects the proper transform $\tilde{D}$ of D transversally. By the discussion above, deformations of this square can be identified with deformations of the triple $(\tilde{Z}, B, \tilde{D})$. The following lemma shows that in this case, the image of α is a neighborhood of $[W]$ in the $\overline{2\text{a}}$ component of $\overline{\mathcal{M}}_{5,5}$. We note that Lemma 5.3 implies that there are no obstructions, so the image of α is smooth.

Theorem 5.6. Let $\mathscr{W} \to T$ be a stable family whose fibers are all 2a or 2b surfaces and $\mathscr{X} \to \mathscr{W}$ be its simultaneous minimal resolution over T, which exists by [KM98, Theorem 7.68]. Then there exists a double cover $j : \mathscr{X} \to \tilde{\mathscr{Z}}$ of smooth schemes over T, where $\tilde{\mathscr{Z}}$ is a smooth family of Hirzebruch surfaces of type $\mathbb{F}_2$ or $\mathbb{F}_0$ blown up at two points on a fiber.

Proof. Let $\psi : \mathscr{X} \to \mathscr{Y}$ be the canonical model of $\mathscr{X}$. Then the canonical map given by the linear system $|\omega_{\mathscr{Y}/T}|$ is a double cover $f : \mathscr{Y} \to \mathscr{Z}$ over T, where fibers of $\mathscr{Z} \to T$ are either smooth or singular quadrics. Let $\mathscr{B}$ denote the branch divisor of f and suppose that $\mathscr{Z}_{t_0}$ is singular for some $t_0 \in T$. Then because the fibers of $\mathscr{X} \to T$ are 2a or 2b surfaces, the branch divisor $\mathscr{B}_{t_0}$ of the map $f|_{t_0}$ is disjoint from the node in $\mathscr{Z}_{t_0}$.

Let $\sigma_1 : \mathscr{Z}_1 \to \mathscr{Z}$ be a simultaneous resolution of singularities of $\mathscr{Z}$ over T. Then the simultaneous resolutions σ_1 and ψ are locally analytically isomorphic in a neighborhood of each singularity of $\mathscr{Z}$, because the branch divisor $\mathscr{B}$ does not intersect the singularities of $\mathscr{Z}$. Thus, no finite base change of T is required to construct $\mathscr{Z}_1$. Letting $\mathscr{Y}_1$ denote the double cover f_1 of $\mathscr{Z}_1$ branched over the preimage $\mathscr{B}_1$ of $\mathscr{B}$, there is a map $\psi_1 : \mathscr{Y}_1 \to \mathscr{Y}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{Y}_1 \arrow[r, "f_1"] \arrow[d, "\psi_1"'] & \mathscr{Z}_1 \arrow[d, "\sigma_1"] \\
\mathscr{Y} \arrow[r, "f"] & \mathscr{Z}
\end{tikzcd}$$

Now let $\mathscr{B}_1$ denote the preimage of $\mathscr{B}$ under σ_1. On each fiber, $\mathscr{B}_1$ has two A_1 singularities. Let S denote the double section of $\mathscr{B}_1 \to T$ passing through these singularities, and let $\sigma_2 : \tilde{\mathscr{Z}} \to \mathscr{Z}_1$ be the blowup of $\mathscr{Z}_1$ in S. Then σ_2 is a simultaneous embedded resolution of singularities of $\mathscr{B}_1$. Thus, no finite base change of T is required to construct $\mathscr{Z}$, and the map $j : \mathscr{X} \to \tilde{\mathscr{Z}}$ defined by $\sigma_1 \circ \sigma_2 \circ j = f \circ \psi$ is a double cover over T. $\square$

Thus, the space of equisingular deformations of W contains a smooth 39-dimensional component corresponding to the closure of the 2a locus in $\overline{\mathcal{M}}_{5,5}$.

5.3 Three technical lemmas

Our goal is to describe the degree two part of the Schouten bracket. We use a method is similar to [Hor75, Hor76a]. In this section, we prove three technical lemmas, analogous to Lemmas 24, 29, and 31 in [Hor75].

Lemma 5.7. The map

$$\zeta_* : H^1(S, T_S(-\log C)) \to H^1(F_1 \amalg F_2, \mathcal{N}_{F_1 \amalg F_2})$$

induced by the surjection $T_S|_{F_1 \amalg F_2} \to \mathcal{N}_{F_1 \amalg F_2}$ is surjective.

Proof. It suffices to show that

$$H^1(\tilde{\mathbb{F}}_2, f_*(T_S(-\log C))) \to H^1(\Delta_0, f_*(\mathcal{N}_{F_1 \amalg F_2}))$$

is surjective. To do this, recall that the surface $\tilde{\mathbb{F}}_2$ admits an action of $\mathbb{Z}/2\mathbb{Z}$ via deck transformations. By Lemma 4.5, the sheaf $f_*(T_S(-\log C))$ decomposes into invariant and anti-invariant eigenspaces as

$$f_*(T_S(-\log C))_+ = T_{\tilde{\mathbb{F}}_2}(-\log(\tilde{D} + \tilde{B})) \quad \text{and} \quad f_*(T_S(-\log C))_- = T_{\tilde{\mathbb{F}}_2}(-\log \tilde{D}) \otimes \mathcal{O}(-\tilde{L}).$$

We have a similar decomposition of $f_*(\mathcal{N}_{F_1 \amalg F_2})$ as follows. By the projection formula, we have

$$f_*(\mathcal{N}_{F_1 \amalg F_2}) = f_*(f^*(\mathcal{N}_{\Delta_0})) = \mathcal{N}_{\Delta_0} \otimes (\mathcal{O}_{\tilde{\mathbb{F}}_2} \oplus \mathcal{O}_{\tilde{\mathbb{F}}_2}(-\tilde{L})).$$

Thus,

$$f_*(\mathcal{N}_{F_1 \amalg F_2}))_+ = \mathcal{N}_{\Delta_0} \quad \text{and} \quad f_*(\mathcal{N}_{F_1 \amalg F_2}))_- = \mathcal{N}_{\Delta_0} \otimes \mathcal{O}(-\tilde{L}) \simeq \mathcal{N}_{\Delta_0}.$$

We show that the maps

$$\zeta_+ : H^1(\tilde{\mathbb{F}}_2, T_{\tilde{\mathbb{F}}_2}(-\log(\tilde{D} + \tilde{B}))) \to H^1(\Delta_0, \mathcal{N}_{\Delta_0})$$

and

$$\zeta_- : H^1(\tilde{\mathbb{F}}_2, T_{\tilde{\mathbb{F}}_2}(-\log \tilde{D}) \otimes \mathcal{O}(-\tilde{L})) \to H^1(\Delta_0, \mathcal{N}_{\Delta_0})$$

are surjective.

To show the first, we have the exact sequence

$$0 \to T_{\tilde{\mathbb{F}}_2}(-\log(\Delta_0 + \tilde{D} + \tilde{B})) \to T_{\tilde{\mathbb{F}}_2}(-\log(\tilde{D} + \tilde{B})) \to \mathcal{N}_{\Delta_0} \to 0$$

and so it suffices to show that $H^2(\tilde{\mathbb{F}}_2, T_{\tilde{\mathbb{F}}_2}(-\log(\Delta_0+\tilde{D}+\tilde{B}))) = 0$. By Serre duality, this is equivalent to the vanishing of $H^0(\tilde{\mathbb{F}}_2, \Omega^1_{\tilde{\mathbb{F}}_2}(\log(\Delta_0+\tilde{D}+\tilde{B}))\otimes\mathcal{O}(K))$. This is the statement of Lemma 5.3.

For the second, note that we have the exact sequence

$$0 \to T_{\tilde{\mathbb{F}}_2}(-\log\tilde{D}+\Delta_0)\otimes\mathcal{O}(-\tilde{L}) \to T_{\tilde{\mathbb{F}}_2}(-\log\tilde{D})\otimes\mathcal{O}(-\tilde{L}) \to \mathcal{N}_{\Delta_0} \to 0.$$

By Lemma 5.4, we have $H^2(T_{\tilde{\mathbb{F}}_2}(-\log\tilde{D})\otimes\mathcal{O}(-\tilde{L})) = \mathbb{C}$. Moreover, $H^2(\mathcal{N}_{\Delta_0}) = 0$, and thus the map ζ_- is surjective as long as $H^2(\tilde{\mathbb{F}}_2, T_{\tilde{\mathbb{F}}_2}(-\log\tilde{D}+\Delta_0)\otimes\mathcal{O}(-\tilde{L})) = \mathbb{C}$. Equivalently, we show that $H^0(\tilde{\mathbb{F}}_2, \Omega^1_{\tilde{\mathbb{F}}_2}(\log\tilde{D}+\Delta_0)\otimes\mathcal{O}(K+\tilde{L})) = \mathbb{C}$.

By Lemma 4.4 and the projection formula, we have

$$\sigma_*\Omega^1_{\tilde{\mathbb{F}}_2}(\log(\tilde{D}+\Delta_0))\otimes\mathcal{O}(K+\tilde{L}) = \Omega^1_{\mathbb{F}_2}(\log(D+\Delta_0))\otimes\mathcal{O}(\Delta)\otimes\mathfrak{M}_{p,q}.$$

So we now want

$$H^0(\mathbb{F}_2, \Omega^1_{\mathbb{F}_2}(\log(D+\Delta_0))\otimes\mathcal{O}(\Delta)\otimes\mathfrak{M}_{p,q}) = \mathbb{C}.$$

We claim that the sheaf $T_{\mathbb{F}_2}(-\log(D+\Delta_0))$ fits into an exact sequence as

$$0 \to \mathcal{O}(G) \to T_{\mathbb{F}_2}(-\log(D+\Delta_0)) \to \pi^*T_{\mathbb{P}^1}(-a) \to 0$$

where G is a divisor on $\mathbb{F}_2$ and $\pi(D) = a \in \mathbb{P}^1$. To see this, note first that $\pi^*T_{\mathbb{P}^1}(-a) \simeq \mathcal{O}_{\mathbb{F}_2}(D)$. Let $U \subset \mathbb{F}_2$ be an open neighborhood of the point $0 \in D\cap\Delta_0$ with coordinates (x, y) so that D has local equation x and Δ_0 has local equation y. Then the map

$$T_{\mathbb{F}_2}(-\log(D+\Delta_0)) \to \mathcal{O}_{\mathbb{F}_2}(D)$$

is locally given by

$$x\frac{\partial}{\partial x} \mapsto x \qquad y\frac{\partial}{\partial y} \mapsto 0.$$

Thus the map is surjective. Since $T_{\mathbb{F}_2}(-\log(D+\Delta_0))$ is a torsion-free vector bundle of rank two and $\mathcal{O}_{\mathbb{F}_2}(D)$ is a line bundle, the kernel of the map $T_{\mathbb{F}_2}(-\log(D+\Delta_0)) \to \mathcal{O}_{\mathbb{F}_2}(D)$ is a torsion-free vector bundle of rank one. All such vector bundles are given by $\mathcal{O}_{\mathbb{F}_2}(G)$

for some divisor G on $\mathbb{F}_2$. We find G by calculating the Chern class of $T_{\mathbb{F}_2}(-\log(D+\Delta_0))$. The determinant line bundle $\bigwedge^2 T_{\mathbb{F}_2}(-\log(D+\Delta_0))$ is given by $-\mathcal{O}(-K_{\mathbb{F}_2} - D - \Delta_0) = \mathcal{O}(\Delta_0 + 3\Gamma)$, so $c_1(T_{\mathbb{F}_2}(-\log(D+\Delta_0))) = \Delta_0 + 3\Gamma$. Thus, $G = \Delta_0 + 2\Gamma$.

Dualizing the above exact sequence and tensoring with $\mathcal{O}(\Delta) \otimes \mathfrak{M}_{p,q}$ results in the exact sequence

$$0 \to \mathcal{O}(\Delta_0 + \Gamma) \otimes \mathfrak{M}_{p,q} \to \Omega^1_{\mathbb{F}_2}(\log D + \Delta_0) \otimes \mathcal{O}(\Delta) \otimes \mathfrak{M}_{p,q} \to \mathcal{O}_{\mathbb{F}_2} \otimes \mathfrak{M}_{p,q} \to 0.$$

The sheaf on the right has no global sections, since the only section of $\mathcal{O}_{\mathbb{F}_2}$ vanishing at p and q is zero. Moreover, since $\Delta_0 \cdot (\Delta_0 + \Gamma) = -1$ every divisor in the linear system $|\Delta_0 + \Gamma|$ is a union of two divisors Δ_0 and Γ. Since there is only one such divisor passing through p and q, namely the divisor $\Delta_0 + D$, we have

$$H^0(\mathbb{F}_2, \Omega^1_{\mathbb{F}_2}(\log D + \Delta_0) \otimes \mathcal{O}(\Delta) \otimes \mathfrak{M}_{p,q}) \simeq H^0(\mathbb{F}_2, \mathcal{O}(\Delta_0 + D) \otimes \mathfrak{M}_{p,q}) = \mathbb{C},$$

as we wished to show. $\square$

A key ingredient of Horikawa's description in [Hor76a] is a map

$$\gamma : H^1(S, T_S) \to H^0(G, \mathcal{O}(K_S|_G)),$$

where $K_S = 2G + F$ and G is a generic fiber of the map $g : S \to \mathbb{P}^1$.

Lemma 5.8. Let S be a smooth surface with a surjective map $g : S \to \mathbb{P}^1$ such that $g_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^1}$ and let G denote a generic fiber of g. Suppose that $K_S = 2G + F$ for some smooth divisor F on S such that $G \not\subset F$ and let

$$\zeta_* : H^1(S, T_S) \to H^1(F, \mathcal{N}_F)$$

be the map induced by the surjection $T_S|_F \to \mathcal{N}_F$. If the irregularity $q(S) = 0$, $h^1(S, \mathcal{O}(G)) = 0$, and $h^0(F, \mathcal{O}(K - G)|_F) = 0$, then there is a map $\gamma : H^1(S, T_S) \to H^0(F, \mathcal{O}(K_S|_F))$, defined below, with the property that $\mathrm{Ker}\ \gamma = \mathrm{Ker}\ \zeta_*$.

Proof. Cover S by open neighborhoods U_i and let κ_{ij}, d_{ij}, and ζ_{ij} denote transition functions for the line bundles $[K]$, $[G]$ and $[F]$, respectively. We can assume that $\kappa_{ij} = d_{ij}^2\zeta_{ij}$. Since $g_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^1}$, we have $h^0(S, \mathcal{O}(G)) = 2$. Let $\{g, h\}$ be a basis of $H^0(S, \mathcal{O}(G))$, represented by holomorphic functions g_i and h_i on U_i such that $g_i = d_{ij}g_j$ and $h_i = d_{ij}h_j$, and let ζ_i be local equations of F on U_i such that $\zeta_i = \zeta_{ij}\zeta_j$.

Let ρ be an element of $H^1(S, T_S)$ represented by a Čech 1-cocycle $\{\rho_{ij}\}$. Given a function $f \in \mathcal{O}(U_i \cap U_j)$, let $\rho_{ij} \cdot f$ denote the action of ρ on f.

There is a map $H^1(S, T_S) \times H^1(S, \Omega^1_S) \to H^2(S, \mathcal{O}_S)$ defined as follows: given an element $\rho \in H^1(S, T_S)$ corresponding to the first order infinitesimal deformation S_ρ and a line bundle $\mathcal{E}$, the cohomology class of the cup product $[\rho \cup c_1(\mathcal{E})] \in H^2(S, T_S \otimes \Omega^1_S)$ corresponds, via the duality pairing $T_S \otimes \Omega^1_S \to \mathcal{O}_S$, to an element of $H^2(S, \mathcal{O}_S)$. We write this element as $[\rho \cdot \mathcal{E}]$. If ξ_{ij} are transition functions for $\mathcal{E}$, then the first Chern class $c_1(\mathcal{E})$ is represented by the one-cocycle $\{\frac{1}{2\pi i} d(\log \xi_{ij})\}$. Thus, up to a multiplicative constant, the element $[\rho \cdot c_1(\mathcal{E})]$ is represented by the Čech 1-cocycle

$$\left\{ \frac{\rho_{jk} \cdot \xi_{ij}}{\xi_{ij}} \right\}. \tag{5.2}$$

The line bundle $\mathcal{E}$ extends over the deformation S_ρ if and only if $[\rho \cdot \mathcal{E}] = 0 \in H^2(S, \mathcal{O}_S)$ [Ser06, Theorem 3.3.11].

Since the line bundle K extends over the first order infinitesimal deformation S_ρ, we have $[\rho \cdot K] = 0 \in H^2(S, \mathcal{O}_S)$. Hence, there is a 1-cochain $\{\nu_{ij}\}$ with coefficients in $\mathcal{O}_S$ such that

$$\nu_{ij} - \nu_{ik} + \nu_{jk} = \frac{\rho_{jk} \cdot \kappa_{ij}}{\kappa_{ij}}. \tag{5.3}$$

Let f be an element of $H^0(S, \mathcal{O}(K))$, represented by holomorphic functions $\{f_i\}$ on U_i such that $f_i = \kappa_{ij} f_j$ on U_{ij}. Using Equation (5.3) together with the fact that $\{\rho_{ij}\}$ is a 1-cocycle, we see that $\{\rho_{ij} \cdot f_i - f_i \nu_{ij}\}$ represents a 1-cocycle with coefficients in $\mathcal{O}(K)$.

Take $f_i = g_i^2 \zeta_i$. Then

$$\{\rho_{ij} \cdot g_i^2 \zeta_i - g_i^2 \zeta_i \nu_{ij}\} = \{2 g_i \zeta_i \rho_{ij} \cdot g_i + g_i^2 \rho_{ij} \cdot \zeta_i - g_i^2 \zeta_i \nu_{ij}\}$$

represents a 1-cocycle with coefficients in $\mathcal{O}(K)$. Dividing through by g_i, we see that

$$\{2\zeta_i\rho_{ij} \cdot g_i + g_i\rho_{ij} \cdot \zeta_i - g_i\zeta_i\nu_{ij}\}$$

represents a 1-cocycle with coefficients in $\mathcal{O}(K - G)$. Since

$$h^1(S, \mathcal{O}(K - G)) = h^1(S, \mathcal{O}(G)) = 0,$$

there exist holomorphic functions α_i on U_i such that

$$d_{ij}\zeta_{ij}\alpha_j - \alpha_i = 2\zeta_i\rho_{ij} \cdot g_i + g_i\rho_{ij} \cdot \zeta_i - g_i\zeta_i\nu_{ij}. \tag{5.4}$$

Similarly, there exist holomorphic functions β_i on U_i such that

$$d_{ij}\zeta_{ij}\beta_j - \beta_i = 2\zeta_i\rho_{ij} \cdot h_i + h_i\rho_{ij} \cdot \zeta_i - h_i\zeta_i\nu_{ij}. \tag{5.5}$$

Multiplying Equation (5.4) by $d_{ij}h_j$ and Equation (5.5) by $d_{ij}g_j$ and subtracting, we have

$$\kappa_{ij}(h_j\alpha_j - g_j\beta_j) - (h_i\alpha_i - g_i\beta_i) = 2\zeta_i(h_i\rho_{ij} \cdot g_i - g_i\rho_{ij} \cdot h_i) \tag{5.6}$$

and we see that $\{(h_i\alpha_i - g_i\beta_i)|_F\}$ represents a holomorphic section of $\mathcal{O}(K|_F)$. This is the definition of $\gamma(\rho)$. We note that this definition is independent of the choice of ν_{ij}, since those terms cancel when we subtract multiples of Equations 5.4 and 5.5 to define γ.

We claim that the definition of $\gamma(\rho)$ is also independent of choice of ρ_{ij}, α_i and β_i. Since γ is linear, it suffices to show that $\gamma(\rho) = 0$ if $[\{\rho_{ij}\}] = 0$ as an element of $H^1(S, T_S)$.

Suppose that there exist holomorphic vector fields η_i on U_i such that $\rho_{ij} = \eta_j - \eta_i$ on U_{ij}. Then since $\{\kappa_{ij}\}$ is a multiplicative 1-cocycle, we have

$$\frac{\rho_{jk} \cdot \kappa_{ij}}{\kappa_{ij}} = -\frac{\eta_j \cdot \kappa_{ij}}{\kappa_{ij}} + \frac{\eta_k \cdot \kappa_{ik}}{\kappa_{ik}} - \frac{\eta_k \cdot \kappa_{jk}}{\kappa_{jk}} \tag{5.7}$$

on U_{ijk}. Recalling the definition of ν_{ij} in Equation (5.3), we have that $\{\nu_{ij} + \frac{\eta_j \cdot \kappa_{ij}}{\kappa_{ij}}\}$ is a 1-cocycle with coefficients in $\mathcal{O}_S$. The cohomology group $H^1(S, \mathcal{O}_S)$ is 0, so there exist holomorphic functions u_i on U_i such that

$$\nu_{ij} + \frac{\eta_j \cdot \kappa_{ij}}{\kappa_{ij}} = u_j - u_i$$

on U_{ij}. Then by definition of η_i, u_i and Equation (5.7), together with the fact that $\kappa_{ij} = d_{ij}^2\zeta_{ij}$, we have

$$\begin{aligned} g_i\rho_{ij} \cdot \zeta_i + 2\zeta_i\rho_{ij} \cdot g_i - g_i\zeta_i\nu_{ij} & \\ &= d_{ij}\zeta_{ij}(g_j\eta_j \cdot \zeta_j + 2\zeta_j\eta_j \cdot g_j - g_j\zeta_j u_j) - (g_i\eta_i \cdot \zeta_i + 2\zeta_i\eta_i \cdot g_i - g_i\zeta_i u_i) \end{aligned}$$

on U_{ij}. Together with Equation (5.4), we see that $\{\alpha_i - g_i\eta_i \cdot \zeta_i - 2\zeta_i\eta_i \cdot g_i + g_i\zeta_i u_i\}$ represents an element of $H^0(S, \mathcal{O}(G+F))$. Thus, $\{\alpha_i - g_i\eta_i \cdot \zeta_i\}$ represents an element of $H^0(F, \mathcal{O}((G+F)|_F)) = H^0(F, \mathcal{O}(K-G)|_F)$. This last cohomology group is zero by assumption. Thus, $\alpha_i = g_i\eta_i \cdot \zeta_i$ on $F \cap U_i$. Similarly $\beta_i = h_i\eta_i \cdot \zeta_i$ on $F \cap U_i$ and so $h_i\alpha_i - g_i\beta_i = 0$, as we wished to show.

We now show that $\mathrm{Ker}\gamma = \mathrm{Ker}\zeta_*$. On $F \cap U_i$, we have $\zeta_*\rho = (\rho_{ij} \cdot \zeta_i)|_F$. Suppose that $\zeta_*\rho = 0$. Then there exist holomorphic functions v_i on U_i such that on $F \cap U_i$ we have

$$\zeta_{ij}v_j - v_i = (\rho_{ij} \cdot \zeta_i)|_F.$$

By definition of α (see Equation 5.4)

$$d_{ij}\zeta_{ij}\alpha_j|_F - \alpha_i|_F = (g_i\rho_{ij} \cdot \zeta_i)|_F.$$

Therefore the collection $\{\alpha_i|_F - g_i v_i\}$ represents a holomorphic section of $\mathcal{O}_F(G+F)|_F$. Since $H^0(F, \mathcal{O}_F(G+F)|_F) = 0$, we see that $\alpha_i|_F = g_i v_i$ on $F \cap U_i$. Similarly, $\beta_i|_F = h_i v_i$ on $F \cap U_i$. Thus, $\gamma(\rho) = 0$.

Conversely, if $\gamma(\rho) = 0$, then $h_i\alpha_i = g_i\beta_i$ on $F \cap U_i$. Since g and h have no common zeros, we can define holomorphic functions v_i on $F \cap U_i$ by

$$v_i = \frac{\alpha_i}{g_i}|_F = \frac{\beta_i}{h_i}|_F.$$

Then by Equation (5.4), we have

$$\zeta_{ij}v_j - v_i = (\rho_{ij} \cdot \zeta_i)|_F.$$

Thus, $\zeta_*(\rho) = 0$. $\square$

Lemma 5.9. With the same hypotheses as Lemma 5.8, if $[\rho, \rho] = 0$ then $(\gamma(\rho))^2$ is in the image of the restriction map $H^0(S, \mathcal{O}(2K)) \to H^0(F, \mathcal{O}(2K|_F))$.

Proof. We follow closely the proof of Lemma 31 in [Hor75].

Let U_i be a sufficiently fine open cover of S and let $z_i = (z_i^1, z_i^2)$ be holomorphic coordinates on U_i such that $z_i = b_{ij}(z_j)$ on U_{ij}, where b_{ij} are holomorphic functions of z_i. Let ρ be an element of $H^1(S, T_S)$ such that $[\rho, \rho] = 0$, and represented by a 1-cocycle $\{\rho_{ij}\}$. Define

$$\tilde{\phi}_{ij} = b_{ij}(z_j) + \rho_{ij} t.$$

Because ρ is a one-cocycle, the $\tilde{\phi}_{ij}$ are local first order deformations of S which glue to give a global first order deformation S_ρ of S. That is,

$$\tilde{\phi}_{ik} - \tilde{\phi}_{ij}(\tilde{\phi}_{jk}, t) \equiv 0 \text{ mod } (t^2). \tag{5.8}$$

The local deformations $\tilde{\phi}_{ij}$ can be extended to the second order as

$$\phi_{ij} = b_{ij}(z_j) + \rho_{ij} t + \tilde{\rho}_{ij} t^2.$$

where $\{\tilde{\rho}_{ij}\}$ is a one-cochain with coefficients in T_S whose Čech differential gives the two-cocycle $\{[\rho_{ij}, \rho_{jk}]\}$.

Since $[\rho, \rho] = 0$, the ϕ_{ij} glue to give a global second order deformation of S. That is

$$\phi_{ik} - \phi_{ij}(\phi_{jk}, t) \equiv 0 \text{ mod } (t^3). \tag{5.9}$$

Since K extends over the second-order deformation given by ϕ_{ij}, we have $[\rho \cdot K] = 0 \in H^2(S, \mathcal{O}_S)$ (see Equation (5.2)). Hence, there is a 1-cochain $\{\nu_{ij}\}$ with coefficients in $\mathcal{O}_S$ such that

$$\nu_{ij} - \nu_{ik} + \nu_{jk} = \frac{\rho_{jk} \cdot \kappa_{ij}}{\kappa_{ij}}. \tag{5.10}$$

Gluing function of the second order deformation satisty

$$\Psi_{ij} = \kappa_{ij} + \kappa_{ij} \nu_{ij} t (\text{ mod} t^2), \tag{5.11}$$

see [Hor75, Lemma 31].

Now let f be a holomorphic section of $\mathcal{O}(K)$ over S, represented by a collection $\{f_i\}$ of holomorphic functions on U_i such that $f_i = \kappa_{ij} f_j$ on U_{ij}. Then, the collection

$$\{\rho_{ij} \cdot f_i - f_i \nu_{ij}\}$$

represents a 1-cocycle with coefficients in $\mathcal{O}(K)$. Thus we can find holomorphic functions τ_i on U_i such that

$$\kappa_{ij}\tau_j - \tau_i = \rho_{ij} \cdot f_i - f_i \nu_{ij}$$

on U_{ij}. Moreover, the functions

$$\Phi_i = f_i(z_i) + \tau_i t$$

give local first order deformations of the section f_i along S_ρ which glue to give a global first order deformation of f. That is

$$\Phi_i(\phi_{ij}, t) - \Psi_{ij}\Phi_j \equiv 0 \bmod (t^2). \tag{5.12}$$

If we define Γ_{ij} to be the homogeneous part of degree two of Equation (5.12), then the Γ_{ij}/t^2 are obstructions to deforming f to the second order along S_ρ. Thus, the collection $\{\Gamma_{ij}/t^2\}$ forms a 1-cocycle with coefficients in $\mathcal{O}(K)$. Since $H^1(S, \mathcal{O}(K)) = 0$, this 1-cocycle is cohomologous to 0.

Let $\{g, h\}$ be a basis of $H^0(S, \mathcal{O}(G))$. We apply the above argument to $f^0 = g^2\zeta$, $f^1 = h^2\zeta$, and $f^2 = gh\zeta$, where ζ is the local equation of F. Let α_i and β_i be solutions to Equation (5.4) and 5.5 in the proof of Lemma 5.8. Then we can choose

$$\tau_i^0 = g_i\alpha_i, \quad \tau_i^1 = h_i\beta_i, \quad \tau_i^2 = (h_i\alpha_i + g_i\beta_i)/2.$$

Define

$$\Phi_i^k = f_i^k + \tau_i^k t$$

for $i = 0, 1, 2$. A straightforward computation shows that

$$(\Phi_i^2)^2 - \Phi_i^0\Phi_i^1 \equiv (h_i\alpha_i - g_i\beta_i)^2 t^2/4 \bmod (t^3). \tag{5.13}$$

On the other hand, using Equations 5.11 and 5.12, we obtain

$$(\Phi_i^2)^2 - \Phi_i^0\Phi_i^1 \equiv \kappa_{ij}^2((\Phi_j^2)^2 - \Phi_j^0\Phi_j^1) + 2f_i^2\Gamma_{ij}^2 - f_i^0\Gamma_{ij}^1 - f_i^1\Gamma_{ij}^0 \bmod (t^3). \tag{5.14}$$

where Γ_{ij}^k denotes the homogeneous degree two part of the left hand side of Equation (5.12) with Φ_i replaced with Φ_i^k.

Combining Equations 5.14 and 5.13, we have

$$(h_i\alpha_i - g_i\beta_i)^2)t^2/4 \equiv \kappa_{ij}^2((\Phi_j^2)^2 - \Phi_j^0\Phi_j^1) + 2f_i^2\Gamma_{ij}^2 - f_i^0\Gamma_{ij}^1 - f_i^1\Gamma_{ij}^0 \bmod (t^3) \tag{5.15}$$

on U_{ij}.

Now, the exact sequence

$$0 \to \mathcal{O}_S(2G+K) \to \mathcal{O}_S(2K) \to \mathcal{O}_F(2K|_F) \to 0$$

gives rise to the exact sequence

$$H^0(S, \mathcal{O}(2K)) \longrightarrow H^0(F, \mathcal{O}(2K|_F)) \xrightarrow{\delta} H^1(S, \mathcal{O}(2G+K)) .$$

By Equations (5.14) (5.15), and the definition of γ (see Equation (5.6)), the cohomology class of $\delta((\gamma(\rho))^2)$ is represented by the 1-cocycle

$$\frac{1}{t^2}(2f_i^2\Gamma_{ij}^2 - f_i^0\Gamma_{ij}^1 - f_i^1\Gamma_{ij}^0).$$

As we saw above, each 1-cocycle $\{\Gamma_{ij}^k/t^2\}$ is cohomologous to 0, and so $(\gamma(\rho))^2$ is a restriction of some element of $H^0(S, \mathcal{O}(2K))$, as we wished to show. □

5.4 Proof of the main theorem

We describe the space $\mathrm{Def}^{QG}(W)$ of $\mathbb{Q}$-Gorenstein deformations of a general 2b surface W.

Lemma 5.10. [Hor76a, Lemma 6.3] Let S be the minimal resolution of a surface of type 2b. Then $h^1(S, \mathcal{O}(G)) = 2$ and $h^1(S, \mathcal{O}(G + F_1 + F_2)) = 0$.

Proof. Horikawa proves this in the case that S is a double cover of $\mathbb{F}_2$ with a smooth branch divisor. The proof uses Riemann-Roch and Serre duality together with the fact that the canonical divisor on S is given by $K_S = 2G + F_1 + F_2$. Because it only relies on numerical characteristics of S, G, F_1 and F_2, Horikawa's proof works in our case as well. $\square$

By Lemma 5.10, we can define the map γ as in Lemma 5.8, where $F = F_1 + F_2$. By abuse of notation, we let

$$\gamma : H^1(S, T_S(-\log C)) \to H^0(F_1 \amalg F_2, \mathcal{O}_{F_1 \amalg F_2})$$

be the restriction of this map to $H^1(S, T_S(-\log C))$. We note that this map is the restriction to $H^1(S, T_S(-\log C)) \subset H^1(S, T_S)$ of the corresponding map defined in [Hor76a] under the assumption that the branch locus is smooth.

Proof of Theorem 5.1. The deformation space $\mathrm{Def}^{QG,\mathrm{e.s}}(W)$ is locally analytically isomorphic to the zero-set of the Kuranishi map

$$k : H^1(S, T_S(-\log C)) \to H^2(S, T_S(-\log C)) = \mathbb{C}.$$

Choose a basis $\rho_1, \rho_2, \ldots, \rho_{40}$ of $H^1(S, T_S(-\log C))$. Let $t_1, t_2, \ldots, t_{40}$ be the dual basis. A priori, the Kuranishi map is some power series in $t_1, \ldots, t_{40}$. However in our case, we know that $\mathrm{Def}^{QG,\mathrm{e.s}}(W)$ contains a smooth 39-dimensional subspace corresponding to deformations of a 2b surface to a 2a surface (see Section 5.2). This implies that if we choose a basis $\rho_1, \rho_2, \ldots, \rho_{40}$ of $H^1(S, T_S(-\log C))$ such that $\rho_1 \in H^1_-(S, T_S(-\log C))$ and $\rho_i \in H^1_+(S, T_S(-\log C))$ for $i > 2$, then the corresponding dual basis has the property that the Kuranishi function factors into at least two terms, one of which has linear term t_1. To show that $\mathrm{Def}^{QG,\mathrm{e.s}}(W)$ is locally a product of two smooth 39-dimensional components meeting transversally in a 38-dimensional component, it therefore suffices to show that the degree two part of the Kuranishi map is nonzero and not a square. The degree two part is given by the Schouten bracket, defined above.

We restrict the Schouten bracket $[,]$ to $H^1(S, T_S(-\log C)) \otimes H^1(S, T_S(-\log C))$. We claim that the Lie bracket $H^2(S, T_S(-\log C) \otimes T_S(-\log C)) \to H^2(S, T_S)$ has image in $H^2(S, T_S(-\log C))$. Let $\{U_i\}$ be a sufficiently fine open covering of S and let $U_{ijk} = U_i \cap U_j \cap U_k$. Let ρ be an element of $H^2(S, T_S(-\log C) \otimes T_S(-\log C))$, represented by a 2-cocycle $\{\rho_{ij} \otimes \rho_{jk}\}$, where $\{\rho_{ij}\}$ is a 1-cocycle with coefficients in $T_S(-\log C)$. Then ρ_{ij} is a vector field that fixes the ideal sheaf of the C. Thus, $\rho_{ij} \otimes \rho_{jk}$ also fixes the ideal sheaf of C. Therefore the Lie bracket $[\rho_{ij}, \rho_{jk}]$ gives a vector field on U_{ijk} which also fixes the ideal sheaf of C. Thus, the form $[,]$ gives a 2-cocycle with coefficients in $T_S(-\log C)$; that is

$$[,] : H^1(S, T_S(-\log C)) \otimes H^1(S, T_S(-\log C)) \to H^2(S, T_S(-\log C)) \simeq \mathbb{C}.$$

Because $\rho_1, \ldots, \rho_{40}$ and $t_1, \ldots t_{40}$ are dual bases, the degree two part of the Kuranishi map k is given by

$$\sum_{1 \leq i,j \leq 40} [\rho_i, \rho_j] t_i t_j.$$

Moreover, because k factors into a product, one term of which has linear term t_1, we have that $[\rho_i, \rho_j] = 0$ for $2 \leq i, j \leq 40$. It therefore suffices to show that $[\rho_1, \rho_1] = 0$ and $[\rho_1, \rho_i]$ is nonzero for some $i > 1$.

Recall that $K_S = 2G + F_1 + F_2$ and consider the exact sequence

$$0 \to \mathcal{O}_S(2G) \to \mathcal{O}_S(K) \to \mathcal{O}_{F_1} \oplus \mathcal{O}_{F_2} \to 0.$$

On S, we have $p_g = 4$ and $h^0(2G) = 3$, so the image of the map $r : H^0(S, \mathcal{O}_S(K)) \to H^0(S, \mathcal{O}_{F_1} \oplus \mathcal{O}_{F_2})$ is one-dimensional. Moreover, the image of r is contained in the "diagonal" in $H^0(S, \mathcal{O}_{F_1} \oplus \mathcal{O}_{F_2}) \simeq \mathbb{C}^2$. That is, if s is a nonzero global section of $\mathcal{O}_{F_1} \oplus \mathcal{O}_{F_2}$ in the image of r, then $s|_{F_1} \neq 0$ and $s|_{F_2} \neq 0$. More precisely, we have the commutative diagram below, where the arrow on the left is an isomorphism and the inclusion of $H^0(\tilde{\mathbb{F}}_2, \Delta_0)$ into $H^0(S, \mathcal{O}_{F_1} \oplus \mathcal{O}_{F_2})$ sends a section to the section of $\mathcal{O}_{F_1} \oplus \mathcal{O}_{F_2}$ whose restrictions to F_1 and F_2 are equal.

$$\begin{tikzcd} H^0(S, \mathcal{O}_S(K)) \arrow[r, "r"] & H^0(S, \mathcal{O}_{F_1} \oplus \mathcal{O}_{F_2}) \\ H^0(\tilde{\mathbb{F}}_2, \Delta_0 + 2\Gamma) \arrow[u, hook, "\simeq"'] \arrow[r] & H^0(\tilde{\mathbb{F}}_2, \Delta_0) \arrow[u, hook] \end{tikzcd}$$

By Lemmas 5.8, 5.10 and 5.7, the map $\gamma : H^1(S, T_S(-\log C)) \to H^0(S, \mathcal{O}_{F_1} \oplus \mathcal{O}_{F_2})$ is surjective. Thus, we can choose $\rho \in H^1(S, T_S(-\log C))$ such that $\gamma(\rho) \neq 0$, and $\gamma(\rho)^2$ is not in the image of r. But then $\gamma(\rho)^2$ is not a restriction of an element of $H^0(S, \mathcal{O}(2K))$, so by Lemma 5.9, we conclude that $[\rho, \rho] \neq 0$. Thus, the Schouten bracket

$$[,] : H^1(S, T_S(-\log C)) \times H^1(S, T_S(-\log C)) \to H^2(S, T_S(-\log C)) \simeq \mathbb{C}$$

is surjective.

Because it is locally given by the composition of the cup product followed by the Lie bracket of vector fields, the Schouten bracket is $\mathbb{Z}/2\mathbb{Z}$-equivariant under the action of $\mathbb{Z}/2\mathbb{Z}$ by deck transformations. By Lemma 5.2, the invariant part of $H^2(S, T_S(-\log C))$ is zero, and so $[\rho_i, \rho_j]$ is nonzero if and only if $[\rho_i, \rho_j]$ is anti-invariant under the action of $\mathbb{Z}/2\mathbb{Z}$. Suppose that $\rho \otimes \eta$ is an element of $H^1(S, T_S(-\log C)) \otimes H^1(S, T_S(-\log C))$, where ρ and η are either both invariant or both anti-invariant. Then $[\rho, \eta]$ is invariant, that is $[\rho, \eta] \in H^2_+(S, T_S(-\log C))$. By Lemma 5.2, this space is zero, so $[\rho, \eta] = 0$. Thus, by choice of basis, $[\rho_i, \rho_i] = 0$ for all i; in particular, $[\rho_1, \rho_1] = 0$.

Now suppose that $\rho \in H^1_+(S, T_S(-\log C))$ is invariant and $\eta \in H^1_-(S, T_S(-\log C))$ is anti-invariant. Then $[\rho, \eta] \in H^2_-(S, T_S(\log C))$ is anti-invariant. Since $[,]$ is surjective, there exists, by choice of basis, $i > 1$ such that $[\rho_1, \rho_i] \neq 0$, completing the proof. $\square$

CHAPTER 6

FUCHSIAN AND ORBIFOLD DOUBLE NORMAL CROSSING SINGULARITIES

The moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ contains boundary divisors δ_i corresponding to irreducible curves of genera $g-i$ and i intersecting transversally in one point. In the absence of obstructions – and a condition on the orbifold double normal crossing which we describe below – analogous divisors in $\overline{\mathcal{M}}_{K^2,\chi}$ correspond to surfaces with two or more components with orbifold double normal crossing singularities. An *orbifold double normal crossing singularity* is locally analytically of the form

$$(xy = 0) \subset \frac{1}{n}(1, -1, a).$$

See Figure 15 for a visualization of an orbifold double normal crossing singularity.

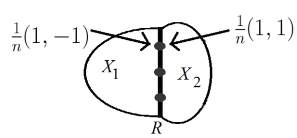


Figure 15. A surface with three orbifold double normal crossing singularities. We see Du Val singularities $\frac{1}{n}(1,-1)$ on X_1, and cyclic quotient singularities $\frac{1}{n}(1,1)$ on X_2.

Let X be a surface with an orbifold double normal crossing singularity such that X consists of two components X_1 and X_2 meeting along a curve $R \subset X$. The divisors $R_1 = R|_{X_1}$ and $R|_{X_2}$ are $\mathbb{Q}$-divisors on X_1 and X_2.

An orbifold double normal crossing singularity has a one-parameter $\mathbb{Q}$-Gorenstein smoothing given by $(xy = tf(z^n)) \subset \frac{1}{n}(1, -1, a, 0)$, where $f(z^n) \in H^0(R, \mathcal{O}_R(R_1|_R +$

$R_2|_R)$). Thus, the equisingular locus will be a divisor in $\overline{\mathcal{M}}_{K^2,\chi}$ as long as $H^0(R, \mathcal{O}_R(R_1|_R + R_2|_R))$ is one-dimensional and there are no obstructions to $\mathbb{Q}$-Gorenstein smoothings. By Riemann-Roch, $H^0(R, \mathcal{O}_R(R_1|_R + R_2|_R)) = \mathbb{C}$ and $H^1(R, \mathcal{O}_R(R_1|_R + R_2|_R)) = 0$ if and only if $\mathcal{O}_R(R_1|_R + R_2|_R)$ is a sufficiently general line bundle on R of degree equal to the genus of R. In the cases we consider, the curve R is rational, so we require the line bundle $R_1 + R_2$ to be of degree 0.

6.1 Weighted homogeneous singularities

The divisor δ_1 in $\overline{\mathcal{M}}_g$ corresponds to curves of genus $g-1$ with an elliptic tail. We expect comparable divisors in $\overline{\mathcal{M}}_{K^2,\chi}$ corresponding to orbifold double normal crossing surfaces, where one of the components is a K3 surface, which are weighted blowups of smoothings of surfaces containing a unique Fuchsian singularity. Fuchsian singularities are a subset of weighted homogeneous singularities, also known as singularities "with a good $\mathbb{C}^*$-action." In this section, we follow primarily Dolgachev [Dol83, Dol96], Looijenga [Loo84], and Pinkham [Pin77b, Pin78] to describe weighted homogeneous singularities and their smoothings.

A weighted homogeneous singularity is locally analytically isomorphic to an affine variety of the form $X = \text{Spec } R$ where $R = \oplus_{i \in \mathbb{Z}} R_i$ is a graded ring. If $R_0 = \mathbb{C}$ and $R_i = 0$ for $i < 0$, the variety X is said to have a *good* $\mathbb{C}^*$-action. Such a variety has a unique fixed point x_0 of the action, corresponding to the maximal ideal $\oplus_{i>0} R_i$.

The simplest example of a variety with a good $\mathbb{C}^*$-action is the cusp $X = (y^2 = x^3) \subset \mathbb{A}^2$. The $\mathbb{C}^*$-action is simply the action $\lambda \cdot (x, y) = (\lambda^2 x, \lambda^3 y)$ for $\lambda \in \mathbb{C}^*$, and since the origin is the unique fixed point, the action is good. To see what is happening algebraically, we note that $X = \text{Spec } R$ where $R = \frac{\mathbb{C}[x,y]}{(y^2 - x^3)}$ is a graded ring where x and y are given weights 2 and 3, respectively.

Given a variety $X = \text{Spec } R$ with a good $\mathbb{C}^*$-action, a smoothing $\mathcal{X}_w = \text{Spec } \mathcal{R} \to \Delta_w$

of X is called a smoothing of negative weight if $\mathcal{X}_w$ has a compatible good $\mathbb{C}^*$-action. Given such a smoothing, we can projectivize by taking $\bar{\mathcal{X}}_w = \text{Proj } \bar{\mathcal{R}}$ where $\bar{\mathcal{R}}_i = \oplus_{l\geq 0}\mathcal{R}_l$. Equivalently, $\bar{\mathcal{X}}_w = \text{Proj}\mathcal{R}[s]$ where s is given weight 1. Let us return to the example of the cusp to see what this means geometrically.

A versal deformation of the cusp of negative weight is the curve $Y = (y^2 = x^3 + ax + b) \subset \mathbb{A}^2$. Projectivizing, we obtain the family $\bar{X}_t = (y^2 = x^3+at^4x+bt^6) \subset \mathbb{P}(2, 3, 1)$. The general fiber is a smooth elliptic curve, while the special fiber $\bar{X}_0$ is the projectivization $\bar{X} \subset \mathbb{P}(2, 3, 1)$ of our original singularity X.

Let us extend this concept to surfaces. A normal affine surface $X = \text{Spec } R$ has *good* $\mathbb{C}^*$*-action* if $R = \oplus_{l\geq 0}R_l$ is a graded algebra and $R_0 = \mathbb{C}$. The unique singularity $x \in X$ corresponds to the maximal ideal $\mathfrak{m} = \oplus_{l>0}R_l$. The following characterization of such surfaces is due to Dolgachev and Pinkham.

Theorem 6.1. [Dol75,Pin77b] A normal affine surface $X = \text{Spec } R$ has a good C^*-action if and only if there exists a simply connected Riemann surface C, a discrete cocompact subgroup $\Gamma \subset \text{Aut}(C)$, and a line bundle L on C to which the action of Γ lifts such that $R \simeq \oplus_{i\geq 0}H^0(C, L^i)^\Gamma$.

Geometrically, X is obtained from the total space of the line bundle $L^\vee$ on C by taking the quotient by Γ and then contracting the zero-section. For instance, if $C = \mathbb{C}$ and Γ is the cyclic group μ_n, then X is a cyclic quotient singularity.

Let $X = \text{Spec } R$ be an affine surface with a good $\mathbb{C}^*$-action, and let $\bar{X} = \text{Proj } R[t]$, where t has weight 1.

Proposition 6.2. [Loo84] The dualizing sheaf $\omega_{\bar{X}}$ is trivial if and only if C is the upper half plane $\mathbb{H}$ and L is the canonical bundle.

A singularity as in Proposition 6.2 is called a *Fuchsian singularity*. Geometrically a Fuchsian singularity is obtained by taking Γ to be the group of orientation-preserving isometries of a tiling of the upper half-plane $\mathbb{H}$ by congruent polygons. See Figure 16 for a visualization.

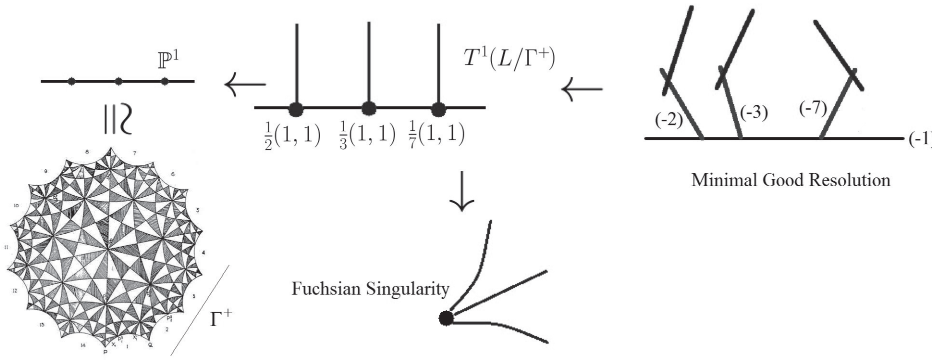


Figure 16. The Fuchsian singularity $D_{2,3,7}$. The triangular tiling has angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$.

6.2 Fuchsian singularities

Let $X = \operatorname{Spec} R$ be a normal affine surface with a unique Fuchsian singularity at x_0 and let $\bar{X} = \operatorname{Proj} R[s]$ be its standard projectivization. Note that all singularities of $\bar{X}$ other than x_0 occur on $\bar{X}_\infty = \bar{X} - X$. If Γ corresponds to an m-gon with angles $\pi/p_1, \pi/p_2, \cdots, \pi/p_m$, then there are m cyclic quotient singularities $\frac{1}{p_1}(1,-1), \frac{1}{p_2}(1,-1), \cdots, \frac{1}{p_m}(1,-1))$ lying on $\bar{X}_\infty$. Note that these singularities are simply Du Val singularities $A_{p_1-1}, A_{p_2}, \cdots, A_{p_m}$.

Let $X' \to X$ be the minimal resolution of the singularities along $\bar{X}_\infty$. Because these are Du Val singularities, the exceptional curve on X' contains the central component and chains of (-2)-curves of lengths $p_1 - 1, \ldots, p_m - 1$.

We denote a Fuchsian singularity with the given minimal resolution by $D_{p_1,\cdots,p_m}(g)$, where g is the genus of the proper transform E of $\bar{X}_\infty$. If $g = 0$ we write simply $D_{p_1,\cdots,p_m}$. When $E^2 = -2$, we represent the exceptional divisor by a graph $T_{p_1,\cdots,p_m}$, where vertices correspond to irreducible curves with self-intersection -2, and edges between vertices exist if the corresponding curves intersect.

Example 6.3. For us, the important examples of $D_{p_1,\cdots,p_m}(g)$ singularities are when $m = 3$. We will see below that in this case $g = 0$. These singularities, denoted by $D_{p,q,r}$, are called *triangle singularities.*

Next, we resolve the Fuchsian singularity of X' to obtain the *minimal good resolution*

$\tilde{X}$ of X, in which the exceptional curves intersect transversally. Figure 16 includes a diagram of the minimal good resolution of a $D_{2,3,7}$ singularity.

Proposition 6.4. [Loo84, OW77] A $D_{p_1,\cdots,p_m}(g)$ singularity exists if and only if $p_i \geq 2$ and $\sum \frac{1}{p_i} < m + 2g - 2$.

Note in particular that $g = 0$ when $m = 3$ and that the triangle singularity $D_{p,q,r}$ exists if and only if $1/p + 1/q + 1/r < 1$. This is obvious geometrically, since triangles with angles $\pi/p, \pi/q, \pi/r$ exist in $\mathbb{H}$ if and only if $1/p + 1/q + 1/r < 1$.

Which Fuchsian singularities are smoothable?

Theorem 6.5. [Pin74] If $X = \text{Spec } R$ has a good $\mathbb{C}^*$-action, then there exists a versal deformation space $\mathcal{X} = \text{Spec } \mathcal{R} \to S = \text{Spec } A$ with a good $\mathbb{C}^*$-action extending that of X.

The following example gives the general idea of the construction of $\mathcal{X}$.

Example 6.6. The triangle singularity $D_{2,3,9}$ is the surface singularity locally analytically isomorphic to

$$X = \mathbb{C}[x, y, z]/(f = x^2 z + y^3 + z^4).$$

The equation f is homogeneous of degree 24 with respect to the $\mathbb{C}^*$-action $\lambda \cdot x = \lambda^9 x$, $\lambda \cdot y = \lambda^8 y$, $\lambda \cdot z = \lambda^6 z$. To construct deformations of $D_{2,3,9}$, we find generators of the Jacobian algebra $\mathbb{C}[x, y, z]/(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$. Explicitly,

$$\mathbb{C}[x, y, z]/(2xz, 3y^2, x^2 + 4z^3) =< 1, x, y, z, xy, yz, z^2, yz^2, z^3, xyz^2 >_{\mathbb{C}}$$

Each of these generators, other than xyz^2, has weight less than 24. Then

$$\begin{aligned} \mathcal{X} &= (x^2 z + y^3 + z^4 + a_1 + a_2 x + a_3 y + a_4 z + a_5 xy + a_6 yz + a_7 z^2 + a_8 yz^2 + a_9 z^3 + a_{10} xyz^2 = 0) \\ &\subset \mathbb{A}^3 \times \text{Spec } A \end{aligned}$$

where $A = \mathbb{C}[a_1, a_2, \ldots, a_{10}]$.

Let $\bar{\mathcal{X}} = \text{Proj}\mathcal{R}[t]$. Note that $\mathcal{R}[t]/(t) \simeq A \otimes_{\mathbb{C}} R$, so that $\bar{\mathcal{X}}_\infty = S \times \bar{X}_\infty$. That is, for $s \in S$, the surface $\bar{X}_s$ contains a curve isomorphic to the original curve $\bar{X}_\infty$.

6.2.1 Deformations of triangle singularities

Let $\pi : \bar{\mathcal{X}} \to S$ be a negative weight deformation of a projectivized surface $\bar{X}$ with a good $\mathbb{C}^*$-action and a unique $D_{p,q,r}$ singularity, as above. As discussed above, every fiber of π contains a divisor isomorphic to $\bar{X}_\infty$ consisting of a configuration of curves corresponding to a $T_{p,q,r}$ graph.

Observe that $H^1(\bar{X}_0, \mathcal{O}_{\bar{X}_0}) = 0$, so by upper semi-continuity, we have $H^1(\bar{X}_s, \mathcal{O}_{\bar{X}_s}) = 0$ for all $s \in S$. Moreover, for $s \neq 0$, there exists a unique nowhere-zero holomorphic two-form ω_s with a $\mathbb{C}^*$-action given by $\lambda \cdot \omega_s = \lambda^{-1}\omega_s$. Therefore, the nonsingular fibers of π are K3 surfaces containing a divisor D_s consisting of (-2) curves whose intersection graph is a $T_{p,q,r}$ graph. Additionally, the family $\{\omega_s\}_{s\in S}$ is a holomorphic family. (Recall that a K3 surface X is a smooth projective surface such that the canonical class ω_X is trivial and $\pi_1(X) = 0$, or equivalently, $H^1(X, \mathcal{O}_X) = 0$.)

The following remarkable theorem of Pinkham tells us that every $K3$ surface X with a $T_{p,q,r}$ curve D such that $X - D$ is affine corresponds to a fiber of π.

Theorem 6.7. [Pin77b,Pin77a], [Loo83, Proposition 2] Let X be a K3 surface containing a $T_{p,q,r}$ curve D such that $X - D$ is affine and let ω be a holomorphic nonzero two-form on X. Then there exists a unique $s \in S$ for which there is a unique isomorphism $X \to X_s$ which maps any irreducible component of D to an irreducible component of D_s. Under this isomorphism, the two-form ω_s pulls back to ω.

The beauty of Theorem 6.7 is that the question of which surfaces are degenerations of $D_{p,q,r}$ singularities is now reduced to finding K3 surfaces endowed with a $T_{p,q,r}$ configuration of curves, which is a purely lattice-theoretic question. Dolgachev and Nikulin give us the answer. Before we state it, let us review some facts about K3 surfaces and lattices.

6.2.2 Some lattice theory for K3 surfaces

Given a lattice S and its dual S^*, there is an injective homomorphism $i_S : S \hookrightarrow S^*$ given by sending x to the function $f_x(y) = x \cdot y$. The discriminant group D_S is defined to be $S^*/i_S(S)$. Note that D_S is a finite group if and only if the intersection pairing is nondegenerate. We say that S is unimodular if $D_S = 0$ and denote by $l(s)$ the minimal number of generators of D_S. If L is another lattice, an embedding $i : S \hookrightarrow L$ is called primitive if $L/i(S)$ is a free group.

Now let X be a K3 surface. Then together with the intersection pairing, the second homology group $H_2(X, \mathbb{Z})$ is isomorphic to the lattice $L = \mathbb{Z}^{22}$. Moreoever, this lattice is even (that is $x \cdot x$ is even for all $x \in \mathbb{Z}^{22}$), unimodular, and of signature $(3, 19)$. Let $Q_{p,q,r}$ denote the lattice $\mathbb{Z}^{p+q+r-2}$ of signature $(1, p+q+r-3)$ with intersection pairing given by the intersection matrix of the $T_{p,q,r}$ diagram. Note that $Q_{p,q,r}$ is even and nondegenerate.

Theorem 6.8. [Loo84] If there exists a primitive embedding $Q_{p,q,r} \hookrightarrow L$, then there is a good smoothing of the $D_{p,q,r}$ singularity.

We note, however, that a good smoothing may exist even if the embedding is not primitive.

Theorem 6.9. [Dol83, Nik79] A primitive embedding of an even nondegenerate lattice S of signature (t_+, t_-) into an even unimodular lattice L of signature (l_+, l_-) exists if

1. $l_+ \geq t_+$,
2. $l_- \geq t_-$, and
3. $\mathrm{rk}(L) - \mathrm{rk}(S) > l(S)$.

In our case,

$$\mathrm{rk}(L) = 22,$$

$$\mathrm{rk}(S) = 1 + p + q + r - 3 = p + q + r - 2.$$

Moreoever, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ and so $p, q, r \geq 2$. Thus, to get a primitive embedding, we need

1. $p + q + r - 3 \leq 19$ and so $p + q + r \leq 22$

2. $22 - (p + q + r - 3) > l(S)$, so $p + q + r < 24 - l(S)$

A priori, we know only that $l(S) \leq \text{rk}(S)$, so

$$22 > l(S) + \text{rk}(S) \geq 2l(S) \Rightarrow l(S) \leq 10.$$

To get a better bound on $l(S)$, we need to calculate the discriminant group D_S.

Let M denote the Gram matrix of S, that is the intersection matrix of $T_{p,q,r}$. Then the size of D_S is given by

$$\det M = pqr(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}).$$

The column span of M corresponds to the image of S under the homomorphism $i : S \hookrightarrow S^*$. Using matrix reduction, one can show that $l(S) \leq 3$. In fact

Lemma 6.10. The discriminant group D_S is generated by 3 elements a, b, c with relations

$$a + b + c = pa = qb = rc.$$

In particular, $l(S) \leq 2$. Moreover, $D_S \equiv \mathbb{Z}/\theta \oplus \mathbb{Z}/\phi$, where θ is the greatest common divisor of p, q, and r, and θ divides ϕ.

Thus, for $p + q + r \leq 21$ (as 21 is strictly less than $24 - l(S)$), we have a primitive embedding of $Q_{p,q,r}$ into the lattice L, and hence there exists a smoothing of the $D_{p,q,r}$ singularity. Wahl [Wah81] showed that if $p + q + r > 22$, then no smoothing exists. The remaining case to consider is thus $p + q + r = 22$. This case was completed by Pinkham [Pin]. To begin with, Pinkham shows that if $p + q + r = 22$, then a primitive embedding exists if the greatest common divisor of p, q and r is 1. The remaining cases to consider are where (p, q, r) is one of $(2, 6, 14)$, $(6, 6, 10)$, or $(2, 10, 10)$. For the first two of these cases, Pinkham constructs an overlattice S in which $\mathbb{Q}_{p,q,r}$ has index two, and shows that S has a primitive embedding into the lattice L. He then proves that a $D_{2,10,10}$ singularity cannot be smoothed. To summarize:

Theorem 6.11. [Dol83, Nik79, Loo83, Pin, Wah81] The Fuchsian singularities $D_{p,q,r}$ which admit a smoothing are those with $p + q + r \leq 22$ and $(p, q, r) \neq (2, 10, 10)$.

There are 22 smoothable Fuchsian singularities $D_{p,q,r}$ that are hypersurface singularities, that is they are locally cut out by a single equation in $\mathbb{A}^3$.

6.3 Hypersurface Fuchsian singularities

Example 6.12. The polynomial $f = x^2z + y^3 + z^4 \in \mathbb{C}[x, y, z]$ is the local equation for the exceptional Fuchsian singularity of type $D_{2,39}$. As described above, this singularity is quasihomogeneous of weight 24 with weights $(9, 8, 6)$. Let $f_1, \dots, f_{10}$ be a basis of

$$\frac{\mathbb{C}[x, y, z]}{(f, \partial_x f, \partial_y f, \partial_z f)}.$$

There is one basis element, say f_{10}, which has "negative weight," i.e. weight greater than 24. Let X_t be the family of quintic surfaces locally analytically given by

$$\{F = f + \sum_{i=1}^{9} a_i t^{i_k} f_i = 0\} \subset \mathbb{P}^3 \times \mathbb{C}_t$$

where i_k is chosen so that F is quasihomogenous of weight 24 with weights $(9, 8, 6, 1)$ as an element of $\mathbb{C}[x, y, z, t]$. By Dolgachev [Dol96], the blowup of X_t with weights $(9, 8, 6, 1)$ has as special fiber a surface with two components S_1 and S_2 meeting along a double curve R of genus 0. The surface S_1 is a K3 surface with three singularities A_1, A_2 and A_8 along $R|_{S_1}$. The surface S_2 has singularities $\frac{1}{2}(1, 1), \frac{1}{3}(1, 1), \frac{1}{9}(1, 1)$ along $R|_{S_2}$. One can check that $R|_{S_1}^2 + R|_{S_2}^2 = 0$.

We calculate the dimension of the moduli space of these surfaces. On the K3 component, resolving the singularities gives an M-polarized K3 surface where M is a lattice of signature $(1, 11)$. The moduli space $\mathbf{K}_M$ of these M-polarized K3 surfaces has dimension 8 [Dol96].

The minimal resolution $\tilde{S}_2$ of S_2 is the minimal good resolution of X_0. By Yang [Yan84], the minimal model Y of $\tilde{S}_2$ is a minimal surface with invariants $K^2 = 2, p_g = 3, q = 0$.

By Horikawa [Hor76a] Y is the double covers of $\mathbb{P}^2$ branched over a curve of degree 8. Moreover, Y contains a cusp with self-intersection -3. The moduli of such surfaces can be identified with the moduli of octic curves in $\mathbb{P}^2$ which are tangent to a cuspidal curve at 12 points. This has dimension 31 and so the locus of stable quintic surfaces arising in this way is 39 dimensional.

Conjecture 6.13. Each of the 22 exceptional Fuchsian singularities corresponds to a (generically) Cartier divisor in $\overline{\mathcal{M}}_{5,5}$.

In his thesis, P. Gallardo [Gal] gives a proof of this conjecture in a number of cases, using a different method. To show smoothness, he uses a theorem of Shustin and Tyomkin [ES99]. In the coming months, I hope to prove Conjecture 6.13 more explicitly by showing that surfaces obtained from smoothings of Fuchsian singularities and containing a K3 component as above have unobstructed $\mathbb{Q}$-Gorenstein deformations.

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